1. Which of the following are rings? (Note: To be a ring, it must have a multiplicative identity.) For those which are not, why not? For those which are, are they commutative? integral domains? fields?

a) $\mathbb{Q}[e^{2\pi i/15}]$.

b) $M_n(\mathbb{Z}[[x]])$.

c) $\mathbb{Z} \times \mathbb{R}$, with $(a,b)+(c,d) = (a+c, b+d)$, $(a,b)\cdot(c,d) = (ac, bd)$.

d) $\mathbb{R} \times \mathbb{R}$, with $(a,b)+(c,d) = (a+c, b+d)$, $(a,b)\cdot(c,d) = (ac-bd, ad+bc)$. (Hint: Where have you seen this before?)

e) $x\mathbb{R}[x]$, the polynomials that are multiples of x.

2. Do the same for the following:

a) $\{$continuous functions $f : \mathbb{R} \to \mathbb{R}\}$ under addition and multiplication.

b) $\{$continuous functions $f : \mathbb{R} \to \mathbb{R}\}$ under addition and composition.

c) $\{$continuous functions $f : \mathbb{R} \to \mathbb{R}$ with compact support$\}$ (i.e. functions that vanish except in some interval $[-n, n]$), under addition and multiplication.

3. Which of the following are ring homomorphisms? For those which are not, why not? For those which are, what are the kernels and images?

a) $f : \mathbb{R}[x] \to \mathbb{R}$, $f(\sum_{i=0}^n a_i x^i) = \sum_{i=0}^n a_i 3^i \quad (a_i \in \mathbb{R})$

b) $f : \mathbb{C} \to \mathbb{C}$, $f(a+bi) = a - bi \quad (a, b \in \mathbb{R})$

c) $f : \mathbb{C} \to \mathbb{C}$, $f(a+bi) = a \quad (a, b \in \mathbb{R})$

d) $f : \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}[\sqrt{3}]$, $f(a + b\sqrt{2}) = a + b\sqrt{3} \quad (a, b \in \mathbb{Z})$

e) $f : \mathbb{Z}[i] \to \mathbb{Z}/n$, $f(a+bi) = a + 2b$. (Hint: Your answer should depend on n.)

4. Let $\mathbb{H} = \{$quaternions $a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$, with multiplication $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$.

a) For $\alpha = a + bi + cj + dk \in \mathbb{H}$, define the conjugate $\bar{\alpha} = a - bi - cj - dk$, and define the absolute value $|\alpha| = \sqrt{a^2+b^2+c^2+d^2}$ (non-negative square root). Show that $|\alpha|^2 = \alpha\bar{\alpha}$ and that $\overline{\alpha\beta} = \bar{\beta}\bar{\alpha}$. Conclude that $|\alpha\beta| = |\alpha||\beta|$. Also, find all α such that $|\alpha| = 0$.

b) Does $\mathbb{H}$ have any (non-zero) zero-divisors? (Hint: Use part (a).)

c) Is $\mathbb{H}$ a division ring? a field?

5. Let $R_1 = \mathbb{R}[[x]]$ = the ring of formal power series $\sum_{i=0}^\infty a_i x^i$ in x, and let $R_2 = \mathbb{R}\{x\}$ = the ring of convergent power series in x (i.e. power series with a positive radius of convergence).

a) Show that R_1 and R_2 are integral domains. [Hint: For $f = \sum_{i=n}^\infty a_i x^i$ (with $n \geq 0$ and $a_n \neq 0$), define the *order* of f to be the integer $\mathrm{ord}(f) = n$. Define $\mathrm{ord}(0) = \infty$. What is $\mathrm{ord}(f \cdot g)$?]

b) Find all units in R_i.

c) Show that the set of non-units is a principal ideal (f) of R_i, for some $f \in R_i$. Find f explicitly, and show that $R_i[1/f]$ is a field.