1. Let R be a commutative ring.

a) Let $I_1, \ldots, I_n$ be ideals in R, and let $\mathfrak{p} \subset R$ be a prime ideal containing $I_1 \cap \cdots \cap I_n$. Show that $I_i \subseteq \mathfrak{p}$ for some i.

b) Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals in R, and let $I \subset R$ be an ideal that is contained in $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$. Show that $I \subseteq \mathfrak{p}_i$ for some i. [Hint: Induction on n.]

c) Explain the content of (a) and (b) geometrically.

2. Let $I_1, \ldots, I_n \subset R$ be ideals in a commutative ring R.

a) Show that $\prod_{j=1}^n I_j = \bigcap_{j=1}^n I_j$ if the ideals are pairwise relatively prime. Explain this assertion geometrically in the case $R = \mathbb{C}[x, y]$.

b) Let $\phi : R \to \prod_{j=1}^n R/I_j$ be the map obtained by reducing modulo each I_j. Must this map be injective? surjective? an isomorphism? Give examples to show the possibilities, and prove a necessary and sufficient condition for ϕ to be an isomorphism.

3. a) Is the Jacobson radical always a radical ideal? Is the nilradical?

b) In each of the following rings, find the Jacobson radical, the nilradical, and the set of units. Also determine if the ring is local. $\mathbb{R}[x, y]$, $\mathbb{R}[[x, y]]$, $\mathbb{R}[x, y]/(y^3)$, $\mathbb{R}[x, y]/(xy)$, $\mathbb{R}[x][[y]]$, $\mathbb{R}[[y]][x]$.

c) Prove that $\mathbb{R}[x]_{(x)} \subset \mathbb{R}[[x]]$, $\mathbb{R}[x, y]_{(x,y)} \subset \mathbb{R}[[x, y]]$, and $\mathbb{Z}_{(p)} \subset \mathbb{Z}_p$, but that $\mathbb{R}[x, y]_{(y)}$ is not a subring of $\mathbb{R}[x][[y]]$. Explain.

4. If $I, J \subseteq R$ are ideals in a commutative ring, define the *ideal quotient* $(I : J) \subseteq R$ to be $\{a \in R \,|\, aJ \subseteq I\}$. Show that this is an ideal. If $R = \mathbb{Z}$, prove that $((m) : (n)) = (m/\gcd(m, n))$.

5. If $R \subseteq S$ are commutative rings and $I \subseteq R$ is an ideal of R, call $IS \subseteq S$ the *extension* of I to S. If $J \subseteq S$ is an ideal of S, call $J \cap R \subseteq R$ the *contraction* of J to R.

a) Are extensions and contractions of ideals always ideals? What about proper ideals? Are extension and contraction inverse operations?

b) For which prime ideals of $\mathbb{Z}$ is the extension to $\mathbb{Z}[i]$ also prime? For those that are not, which extensions are the product of two distinct prime ideals, and which are the square of a prime ideal of $\mathbb{Z}[i]$? (Of these two cases, the former case is called *split* and the latter case is called *ramified*.)

c) Show that taking contraction induces a surjection from the prime ideals of $\mathbb{Z}[i]$ to the prime ideals of $\mathbb{Z}$. Is it injective?

d) Do your assertions in part (c) hold for an arbitrary extension of integral domains $R \subseteq S$?