Math 602 Problem Set #11 Due Wed., Dec. 1, 2021

1. a) If V and W are vector spaces over a field K, and if $F : V \to W$ is a homomorphism, let $F^* : W^* \to V^*$ be the map on dual spaces given by $F^*(\phi) = \phi \circ F$. Show that $F \mapsto F^*$ defines a homomorphism $\mathrm{Hom}(V, W) \to \mathrm{Hom}(W^*, V^*)$. Show that this homomorphism is natural, in the sense that $(F \circ G)^* = G^* \circ F^*$ if $F : V \to W$ and $G : U \to V$.

b) Show that the above map $\mathrm{Hom}(V, W) \to \mathrm{Hom}(W^*, V^*)$ is an isomorphism if V and W are finite dimensional.

c) Show that if $0 \to U \to V \to W \to 0$ is exact, then so is $0 \to W^* \to V^* \to U^* \to 0$.

d) What if instead we consider modules over a ring R?

2. For any finite dimensional vector space V with basis $B = \{e_1, \ldots, e_n\}$, and dual basis $B^* = \{\delta_1, \ldots, \delta_n\}$ of V^*, define $\phi_{V,B} : V \to V^*$ by $\sum_1^n a_i e_i \mapsto \sum_1^n a_i \delta_i$, and let $\psi_{V,B} = \phi_{V^*,B^*} \circ \phi_{V,B}$.

a) Show that $\phi_{V,B} : V \to V^*$ is an isomorphism, but that it depends on the choice of B.

b) Show that $\psi_{V,B} : V \to V^{**}$ is an isomorphism that is independent of the choice of B (so we may denote it by ψ_V). For $v \in V$, show that $\psi_V(v)$ is the element of V^{**} taking $f \in V^*$ to $f(v)$.

c) Show that the association $V \mapsto \psi_V$ is natural in the following sense: If $F : V \to W$ is a vector space homomorphism with induced homomorphisms $F^* : W^* \to V^*$ and $F^{**} : V^{**} \to W^{**}$ (notation as in problem 1), then $\psi_W \circ F = F^{**} \circ \psi_V$.

3. Let V, W, Y be finite dimensional vector spaces over K.

a) Show that there are natural isomorphisms $(V \otimes W)^* = V^* \otimes W^* = \mathrm{Hom}(V, W^*) = \mathrm{Hom}(W, V^*)$.

b) Show that there is a natural isomorphism $\mathrm{Hom}(V \otimes W, Y) = \mathrm{Hom}(V, \mathrm{Hom}(W, Y))$.

c) Show that $\mathrm{Hom}(V \otimes W, Y)$ is naturally isomorphic to the vector space of bilinear maps $V \times W \to Y$.

4. a) Let V be a K-vector space and let $0 \to W' \to W \to W'' \to 0$ be an exact sequence of K-vector spaces. Show that the induced sequences $0 \to V \otimes W' \to V \otimes W \to V \otimes W'' \to 0$; $0 \to \mathrm{Hom}(V, W') \to \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W'') \to 0$; and $0 \to \mathrm{Hom}(W'', V) \to \mathrm{Hom}(W, V) \to \mathrm{Hom}(W', V) \to 0$ are also exact. [Hint: Choose a basis for V.]

b) What if instead we consider modules over a ring R?

5. Let V be a finite dimensional vector space over a field K of characteristic zero, and let $T \in \mathrm{End}\, V$. A subspace $W \subseteq V$ is T-*invariant* if $T(W) \subseteq W$; and W is T-*irreducible* if it is T-invariant and the only T-invariant subspaces of W are 0 and W.

a) Suppose that $T \in \mathrm{End}(V)$ has order n in $\mathrm{End}(V)$ under composition, and that $W \subseteq V$ is a T-invariant subspace. Show that W has a T-invariant complement W'. [Hint: Pick an arbitrary complement W'', i.e. $V = W \times W''$. Let $P : V = W \times W'' \to W$ be the first projection map, and define $S : V \to V$ by $v \mapsto \frac{1}{n} \sum_{i=0}^{n-1} T^i P T^{-i}(v)$. Show $S^2 = S$. Then consider $\ker S$ and $\mathrm{im}\, S$.]

b) Under the hypotheses of (a), show that V can be written as the direct product of T-irreducible subspaces.

c) What if T does *not* have finite order in $\mathrm{End}(V)$?

d) What if K does *not* have characteristic zero?