Math 603 Problem Set #2 Due Mon., Jan. 31, 2005

1. Suppose that

$$\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
\end{array}$$

is a commutative diagram of R-modules, with exact rows.

a) Show that if α_1 is surjective and α_2, α_4 are injective, then α_3 is injective.

b) Show that if α_5 is injective and α_2, α_4 are surjective, then α_3 is surjective.

c) In particular, deduce that α_3 is an isomorphism provided that $\alpha_1, \alpha_2, \alpha_4, \alpha_5$ are.

(The above result is the strong version of the "Five Lemma", which is named after the appearance of this diagram.)

2. In the notation of problems 4 and 5 of Problem Set 1:

a) Show that $I \cap J = (y-4)$ and $I + J = (1)$ in R.

b) Let $\Delta : I \cap J \to I \oplus J$ be given by $\Delta(f) = (f, f)$, and let $- : I \oplus J \to I + J$ be given by $-(f, g) = f - g$. Show that the sequence

$$0 \to I \cap J \xrightarrow{\Delta} I \oplus J \xrightarrow{-} I + J \to 0 \qquad (*)$$

is exact.

c) Show that the exact sequence (*) is split. (Hint: What is $I + J$?)

d) Deduce that $I \oplus J \approx (I \cap J) \oplus (I + J)$, and conclude that $I \oplus J$ is therefore free of rank 2 (thereby giving another proof of problem 5(c) of Problem Set 1).

3. In the notation of the above problem:

a) Explicitly find a section s of $-$, corresponding to a splitting of the exact sequence (*). (Hint: Find $(a, b) \in I \oplus J$ such that $a - b = 1 \in R$.)

b) Explicitly find an isomorphism $\alpha : (I \cap J) \oplus (I + J) \to I \oplus J$ induced by the section in (a).

c) Compare this isomorphism $\alpha : (y-4) \oplus (1) \to I \oplus J$ to the one in problem 5 of Problem Set 1.

4. Let P be a finitely generated projective R-module.

a) Show that there is a finitely generated free R-module F, and an R-module K, such that $0 \to K \xrightarrow{i} F \xrightarrow{\pi} P \to 0$ is exact.

b) Show that there exists a homomorphism $j : F \to K$ such that i is a section of j, and that the sequence $F \xrightarrow{i \circ j} F \xrightarrow{\pi} P \to 0$ is exact.

c) Conclude that P is a finitely presented R-module.

d) Give an example of a finitely generated R-module M (for some R) that is *not* projective and is not even finitely presented.