Math 603 Problem Set #3 Due Fri., Feb. 11, 2005

1. a) Show that $R[x]$ is a flat R-module.
   b) Show that $R[x, y]/(xy)$ is *not* a flat $R[x]$-module.
   c) Let M, N be flat R-modules. Show that $M \oplus N$ and $M \otimes_R N$ are flat R-modules.
   d) Show that if M is a finitely generated projective R-module, then M is a flat R-module.
   e) Is the $\mathbb{Z}$-module $\mathbb{Q}$ free? torsion free? flat? projective?

2. Suppose that $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is an exact sequence of R-modules. Let $M_1 \subset M_2 \subset \cdots$ be a chain of submodules of M, and define $M_i' = f^{-1}(M_i)$ and $M_i'' = g(M_i)$.
   a) Show that $M_1' \subset M_2' \subset \cdots$ is a chain of submodules of M'.
   b) Show that $M_1'' \subset M_2'' \subset \cdots$ is a chain of submodules of M''.
   c) Show that if $i < j$, then the inclusion map $M_i \hookrightarrow M_j$ induces inclusions $M_i' \hookrightarrow M_j'$ and $M_i'' \hookrightarrow M_j''$ and also the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_i' & \longrightarrow & M_i & \longrightarrow & M_i'' & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & M_j' & \longrightarrow & M_j & \longrightarrow & M_j'' & \longrightarrow & 0 \end{array}$$

3. In the notation of Problem Set 1, problems 4 and 5:
   a) Find linear polynomials $f, g \in R$ such that the only maximal ideal of R containing f is I, and the only maximal ideal of R containing g is J. (Hint: Where can the graphs of $f = 0$ and of $g = 0$ intersect the circle?)
   b) Find a linear polynomial $h \in R$ such that $h \in J$ and $h \in K$, where K is the maximal ideal corresponding to the point $S = (3, -4)$.
   c) Show that $I_f \overset{\text{def}}{=} I \otimes_R R[\frac{1}{f}]$ is a free $R[\frac{1}{f}]$-module, viz. is the unit ideal in $R[\frac{1}{f}]$. (Hint: Show $f \in I_f$.)
   d) Show that for suitable choice of g, h above, $\dfrac{x-3}{y-4} = \dfrac{h}{g}$ in R. Explain this equality geometrically, in terms of the graphs of $g = 0$, $h = 0$, $x - 3 = 0$, $y - 4 = 0$, and $x^2 + y^2 = 25$.
   e) Using (d), show that $I_g \overset{\text{def}}{=} I \otimes_R R[\frac{1}{g}]$ is a free $R[\frac{1}{g}]$-module, viz. is the ideal $(y - 4)$ in $R[\frac{1}{g}]$.

4. a) Let R be a commutative ring and suppose that *every* R-module M is free. Show that R is a field.
   b) Let $R = \mathbb{R}[x, y]/(x^2 + y^2 - 25)$. Is R a PID? Is every finitely generated projective R-module free?

5. Let R be a commutative ring, and let M, N, S be R-modules. Assume that M is finitely presented and that S is flat. Consider the natural map

$$\alpha : S \otimes_R \text{Hom}(M, N) \to \text{Hom}(M, S \otimes_R N)$$

taking $s \otimes \phi$ (for $s \in S$ and $\phi \in \mathrm{Hom}(M, N)$) to the homomorphism $m \mapsto s \otimes \phi(m)$.

a) Show that if M is a free R-module then α is an isomorphism. [Hint: If $M = R^n$, show that both sides are just $(S \otimes_R N)^n$.]

b) Suppose more generally that $R^a \to R^b \to M \to 0$ is a finite presentation for M. Show that the induced diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & S \otimes_R \mathrm{Hom}(M, N) & \longrightarrow & S \otimes_R \mathrm{Hom}(R^b, N) & \longrightarrow & S \otimes_R \mathrm{Hom}(R^a, N) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathrm{Hom}(M, S \otimes_R N) & \longrightarrow & \mathrm{Hom}(R^b, S \otimes_R N) & \longrightarrow & \mathrm{Hom}(R^a, S \otimes_R N) \end{array}$$

is commutative and has exact rows.

c) Using the Five Lemma and part (a), deduce that α is an isomorphism.

6. Let $\phi : \mathbb{Z}^2 \to \mathbb{Z}^3$ be given by the matrix

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

and let M be the cokernel of ϕ.

a) Find all $n \in \mathbb{Z}$ such that the $\mathbb{Z}$-module M has (non-zero) n-torsion.

b) Is M free? flat? torsion free? projective?

c) Show that M has a finite free resolution.

d) For each prime number p, compute $M \otimes \mathbb{Z}/p = \mathrm{Tor}^0(M, \mathbb{Z}/p)$ and $\mathrm{Tor}^1(M, \mathbb{Z}/p)$.

e) For every $\mathbb{Z}$-module N and every $i \geq 2$, compute $\mathrm{Tor}^i(M, N)$.

7. Let M, N be R-modules, and let $0 \to N \to I_0 \xrightarrow{f_0} I_1 \xrightarrow{f_1} I_2 \xrightarrow{f_2} \cdots$ be an injective resolution of N. Let $\phi \in \mathrm{Ext}^1(M, N)$, and choose a homomorphism $\Phi \in \mathrm{Hom}(M, I_1)$ representing ϕ (where we use the above resolution to compute Ext).

a) Show that $f_1 \circ \Phi = 0$, and deduce that $\Phi : M \to I_0/N$.

b) Let $M' \to I_0$ be the pullback of $M \to I_0/N$ via the reduction map $I_0 \to I_0/N$. Show that the kernel of $M' \to M$ is N, giving an exact sequence $0 \to N \to M' \to M \to 0$, which corresponds to some class in $\mathrm{Ext}(M, N)$ that we denote by $c(\phi)$. (Here $\mathrm{Ext}(M, N)$ is the set of equivalence classes of extensions $0 \to N \to L \to M \to 0$ of M by N.)

c) Show that $c : \mathrm{Ext}^1(M, N) \to \mathrm{Ext}(M, N)$ is a bijection.