1. a) Show that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules, then M is Noetherian if and only if M' and M'' are. [Hint: For the reverse implication, consider problem 2 of Problem Set 3 and use the Five Lemma.]

b) Deduce that R-modules N_1 and N_2 are each Noetherian if and only if $N_1 \oplus N_2$ is.

c) Let S be a multiplicative subset of a commutative ring R. Show that if R is Noetherian then so is the localization $S^{-1}R$. Does the converse also hold?

2. a) Do there exist infinite strictly increasing chains $I_1 \subset I_2 \subset \cdots$ of ideals in $\mathbb{C}[x]$? Do there exist finite strictly increasing chains $I_1 \subset I_2 \subset \cdots \subset I_n$ of ideals in $\mathbb{C}[x]$ with arbitrarily large n?

b) Repeat part (a), but with strictly *decreasing* chains of ideals $I_1 \supset I_2 \supset \cdots$ rather than strictly increasing chains.

c) Explain geometrically your assertions in parts (a) and (b).

3. Let M and N be finitely generated modules over a commutative ring R, such that $M \otimes_R N = 0$.

a) Show that if R is a local ring with maximal ideal $\mathfrak{m}$, then M or N is 0. [Hint: Nakayama's Lemma.]

b) What if R is not local?

4. Let M be an R-module and let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of R-modules. Under each of the following four conditions (considered separately), either show that the sequence $0 \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$ must be exact or else give a counterexample.

a) M is flat.

b) N' is flat.

c) N is flat.

d) N'' is flat.

5. Let R be a commutative ring with Jacobson radical J. For $x \in R$, let P_x be the set of $r \in R$ such that $r \equiv 1 \pmod{x}$ (i.e. such that $r - 1 \in xR$). Let R^* denote the multiplicative group of units in R.

a) Show that $J = \{x \in R \mid P_x \subset R^*\}$.

b) Let M be a finitely generated R-module and let $a_1, \ldots, a_n \in M$. Show that the R-module M is generated by $a_1, \ldots, a_n$ if and only if the R/J-module M/JM is generated by the images of these elements. [Hint: Either generalize the proof of the local case, or else reduce to that case.]

6. Let M be a finitely generated R-module. Prove that the following conditions on M are equivalent:

i) M is locally free over R (i.e. $M_{\mathfrak{m}}$ is free over $R_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m} \subset R$).

ii) For every maximal ideal $\mathfrak{m} \subset R$, there is an $f \notin \mathfrak{m}$ such that M_f is free over R_f.

iii) $\exists f_1, \ldots, f_n \in R$ such that $(f_1, \ldots, f_n) = 1$ and M_{f_i} is free over R_{f_i} for $i = 1, \ldots, n$.

[Hint: In (ii) $\Rightarrow$ (iii), what ideals contain the set $S = \{f \in R \mid M_f \text{ is free over } R_f\}$?

(over)

7. Let M be an R-module. Prove that the following conditions on M are equivalent:

i) M is faithfully flat (i.e. for every sequence of R-modules $N' \to N \to N''$, the given sequence is exact iff $M \otimes N' \to M \otimes N \to M \otimes N''$ is exact).

ii) For every *complex* of R-modules $N' \to N \to N''$, the given complex is exact iff $M \otimes N' \to M \otimes N \to M \otimes N''$ is exact

iii) M is flat; and for every homomorphism of R-modules $\phi : N_1 \to N_2$, ϕ is surjective iff $1 \otimes \phi : M \otimes N_1 \to M \otimes N_2$ is surjective.

8. Let A be a flat R-algebra. Prove that the following conditions on A are equivalent:

i) A is faithfully flat over R.

ii) Every maximal ideal of R is the contraction of a maximal ideal of A.

iii) $\operatorname{Spec} A \to \operatorname{Spec} R$ is surjective.

iv) For every R-module N, if $A \otimes_R N = 0$ then $N = 0$.

[Hint: For (i) $\Rightarrow$ (ii) and for (iv) $\Rightarrow$ (i), use condition (iii) of problem 7. Also in (i) $\Rightarrow$ (ii), what is the extension to A of a maximal ideal of R? For (ii) $\Rightarrow$ (iii), localize at prime ideals of R. For (iii) $\Rightarrow$ (iv), reduce to the case that N is generated by just one element.]

9. For each of the following R-algebras A, determine whether A is a finitely generated R-module and whether it a finitely generated R-algebra. Also determine whether the R-module A is flat and whether it is faithfully flat.

a) $R = \mathbb{Z}$, $A = \mathbb{Z}[x]/(3x)$.

b) $R = \mathbb{Z}$, $A = \mathbb{Z}[1/5]$.

c) $R = \mathbb{Z}$, $A = \mathbb{Z}[i]$.

d) $R = \mathbb{Z}$, $A = \mathbb{Z}[i, 1/5]$.

e) $R = \mathbb{Z}$, $A = \mathbb{Z}[i, 1/(2+i)]$.

f) $R = \mathbb{R}[x]$, $A = \mathbb{R}[[x]]$

g) $R = \mathbb{R}[x]_{(x)}$, $A = \mathbb{R}[[x]]$.