1. a) Let V be an affine variety, with ring of functions R. Let W be a Zariski closed subset of V, and let $I = I(W)$. Show that W is irreducible if and only if I is a prime ideal.

b) Let $I_1, \dots, I_n$ be proper ideals of R, and let $W_i = V(I_i)$ for each i. Show that

$$V(I_1 + \cdots + I_n) = W_1 \cap \cdots \cap W_n,$$

$$V(I_1 \cap \cdots \cap I_n) = W_1 \cup \cdots \cup W_n.$$

Also explain the relationship with problem 8 on Math 602 (fall 2004) Problem Set 8.

2. a) Let R be a Noetherian ring and $I \subset R$ an ideal. Prove that there are only finitely many prime ideals that are minimal over I. [Hint: If not, show that there is a maximal counterexample I, and that this I is not prime. Show that if $a, b \in R - I$ with $ab \in I$, then every prime that is minimal over I is also minimal over either $I + (a)$ or $I + (b)$.]

b) Deduce that every Noetherian ring has finitely many minimal primes. Also, interpret this assertion geometrically, if R is the ring of functions on a Zariski closed subset of an affine variety V.

c) What happens in (a) and (b) if the ring is not Noetherian?

3. Determine the Krull dimensions of the following rings: $\mathbb{R}[x, x^{-1}]$, $\mathbb{C}[x, y, z]/(z^2 - xy)$, $\mathbb{Z}[x, y]/(y^2 - x^3)$, $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$, $\mathbb{Q}[x, y, z]/(y^2, z^3)$, $\mathbb{Q}[[x, y, z]]$, $\mathbb{Z}_{(2)}[x]$. Justify your assertions.

4. Given a commutative ring R, define the maximal spectrum of R (denoted $\operatorname{Max} R$) to be the set of maximal ideals of R. For each subset $E \subset R$, let $V(E)$ denote the set $\{\mathfrak{m} \in \operatorname{Max} R \mid E \subset \mathfrak{m}\} \subset \operatorname{Max} R$.

a) Show that $\operatorname{Max} R$ has a topology in which the closed sets are precisely the sets $V(E)$.

b) Show that $V(E) = V(I)$ for any $E \subset R$, where I is the ideal generated by E.

c) Show that $V(I) = V(\sqrt{I})$ for any ideal I. (Recall that $\sqrt{I}$ denotes the *radical* of I, which is defined to be $\{r \in R \mid (\exists n)\ r^n \in I\}$.)

d) Show that $V(\bigcup_\alpha E_\alpha) = \bigcap_\alpha V(E_\alpha)$ for any collection of subsets $\{E_\alpha\}_{\alpha \in A}$, and that $V(I_1 + \cdots + I_n) = V(I_1) \cap \cdots \cap V(I_n)$ for any ideals $I_1, \dots, I_n$.

e) Show that $V(I_1 \cap \cdots \cap I_n) = V(I_1) \cup \cdots \cup V(I_n)$ for any ideals $I_1, \dots, I_n$ of R. Also explain the relationship with problem 1 above.

f) Give examples to illustrate (b) - (e) geometrically, in the case $R = \mathbb{R}[x]$, and in the case $R = \mathbb{Z}$.

g) If $R = \mathbb{C}[x, y]/(f)$, is there a continuous bijective map between $\operatorname{Max} R$ and the locus of zeroes of f in $\mathbb{C}^2$ (under the usual topology)? In which direction?

5. Consider the following rings: $\mathbb{C}[x]$, $\mathbb{C}[x, y]$, $\mathbb{C}[x, y]/(x^2 + y^2 - 1)$, $\mathbb{C}[x, y]/(x^2 - y^2)$, $\mathbb{C}[x]/(x^2)$, $\mathbb{C}[x, y]/(x^2)$, $\mathbb{C}$, $\mathbb{C} \times \mathbb{C}$, $\mathbb{C}[x]/(x^2 - x)$, $\mathbb{Z}/2$, $\mathbb{Z}/6$, $\mathbb{Z}$, $\mathbb{Z}[1/15]$. For each of them, do the following:

a) Describe all the maximal ideals in the given ring R, and describe $\operatorname{Max} R$ geometrically (or topologically).

b) Determine whether $\operatorname{Max} R$ is connected (in the topology given in problem 4).

6. a) Let $R = \mathbb{C}[x, y]/(x^2 - y^2)$ and $S = \mathbb{C}[x, y]/(x^2 - x)$. Show that there is a homomorphism $f : R \to S$ given by $f(x) = y - 2xy$, $f(y) = y$. Show that there is an induced continuous map $f^* : \operatorname{Max} S \to \operatorname{Max} R$ given by $\mathfrak{m} \mapsto f^{-1}(\mathfrak{m})$. Describe the map f^* geometrically. Is it injective? surjective? (A picture in the (x, y)-plane may help.)

b) In general, if $f : R \to S$ is a homomorphism of commutative rings, is there an induced continuous map $f^* : \operatorname{Max} S \to \operatorname{Max} R$? (What if $R = \mathbb{Z}$ and $S = \mathbb{Q}$?) What if we instead considered the *prime spectrum* of R and of S? (The prime spectrum $\operatorname{Spec} R$ is defined as the set of prime ideals of R with the topology defined similarly to that of Max.)