Math 603 Problem Set #6 Due Mon., March 14, 2005

1. a) Let R be a Noetherian ring, $\mathcal{I}$ the set of ideals of R, and $\mathcal{I}_0$ a subset of $\mathcal{I}$. Let P be a property that ideals in $\mathcal{I}_0$ may or may not have. Suppose that one can show the following condition:

$$\forall I \in \mathcal{I}_0, \text{ if every ideal } J \in \mathcal{I}_0 \text{ that properly contains } I \text{ has property } P, \text{ then so does } I.$$

Conclude that P holds for *all* $I \in \mathcal{I}_0$.

b) Use this principle ("Noetherian induction") to prove that if R is a Noetherian integral domain, and $r \in R$ is a non-zero non-unit, then r is a product of irreducible elements of R. [Hint: What is $\mathcal{I}_0$?]

c) Show that (b) (and therefore (a)) fails in general if R is not Noetherian.

2. Let $f : Y \to X$ be a polynomial map of complex affine varieties, corresponding to a ring extension $i : A \hookrightarrow B$. Suppose that B is an integral extension of A. Show that the map f is closed in the Zariski topology (where a map is defined to be *closed* if it takes closed sets to closed sets).

3. a) Let $A = \mathbb{C}[x]$, $B = \mathbb{C}[x,y]/(xy-1)$. Is B integral over A? Describe the corresponding map on varieties. Is it closed?

b) Do the same with B replaced by $\mathbb{C}[x,y]/(x^2+y^2-1)$.

c) Do the same with B replaced by $\mathbb{C}[x,y]/(y^2-x)$.

d) Do the same with B replaced by $\mathbb{C}[x,y,\frac{1}{y-1}]/(y^2-x)$.

4. Let n be a square-free non-zero integer. Let R_n be the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{n})$. Show that $R_n = \mathbb{Z}[\frac{1+\sqrt{n}}{2}]$ if $n \equiv 1 \pmod 4$, and that $R_n = \mathbb{Z}[\sqrt{n}]$ otherwise.

5. For each of the following rings R, determine whether R has a height one prime that is not principal. If there is one, find one explicitly. If there isn't one, determine whether there is *some* prime ideal that is not principal, and find one explicitly if it exists.

a) $\mathbb{Z}[i,x,y]$.

b) $\mathbb{Q}[x,y,z,w]/(xy-zw)$.

c) $\mathbb{Z}[\sqrt{-5}]$.

d) $\mathbb{Z}[x,y]/(5, y-x^3-x+1)$.