Math 603 Problem Set #7 Due Fri., March 25, 2005

1. Which of the following rings R are discrete valuation rings? For those that are, find the fraction field $K = \operatorname{frac} R$, the residue field $k = R/\mathfrak{m}$ (where $\mathfrak{m}$ is the maximal ideal), and a uniformizer π. For the others, explain why not (full proofs not required). $\mathbb{Z}$, $\mathbb{Z}_{(5)}$, $\mathbb{Z}[1/5]$, $\mathbb{R}[x]$, $\mathbb{R}[x]_{(x-2)}$, $\mathbb{R}[x, 1/(x-2)]$, $\mathbb{Q}[x]_{(x^2+1)}$, $\mathbb{C}[x,y]_{(x,y)}$, $\big(\mathbb{R}[x,y]/(x^2+y^2-1)\big)_{(x-1,y)}$, $\big(\mathbb{R}[x,y]/(y^2-x^3)\big)_{(x,y)}$.

2. Let R be a discrete valuation ring with fraction field K, maximal ideal $\mathfrak{m}$, and discrete valuation v. If $a, b \in K$ define $\rho(a,b) = 2^{-v(a-b)}$ if $a \neq b$, and define $\rho(a,a) = 0$.

a) Show that ρ defines a metric on K.

b) Show that ρ is an ultrametric (=non-archimidean metric); i.e. it satisfies the strong triangle inequality $\rho(a,c) \le \max(\rho(a,b), \rho(b,c))$.

c) Show that (K, ρ) is a topological field, i.e. that it is a topological space in which addition and multiplication define continuous maps $K \times K \to K$.

d) Show that in K, the closed unit disc about 0 is R and the open unit disc about 0 is $\mathfrak{m}$.

3. Let K be a field and let $f(x) \in K[x]$ be a non-zero polynomial of degree n.

a) Show that if $a \in K$ is a root of f, then $(x-a)$ divides $f(x)$ in $K[x]$. [Hint: Use the division algorithm for polynomials.]

b) Deduce that f has at most n roots in K.

c) Will the argument and conclusion of part (b) still hold if K is replaced by a division algebra (i.e. if K is no longer assumed commutative)? Explain. [Hint: Try an example.]

4. Let R be a commutative ring of characteristic p (where p is prime) and define $F : R \to R$ by $a \mapsto a^p$.

a) Show that F is a ring endomorphism (i.e. homomorphism from R to itself).

b) If R is a field, determine which elements lie in the set $\{a \in R \mid F(a) = a\}$.

c) If R is a field, must F be injective? surjective? (Give a proof or counterexample for each.)

d) If R is a finite field, show that F is an automorphism.

5. Let K be a field and let G be a subgroup of the multiplicative group $K^* = K - \{0\}$.

a) Show that if $a, b \in K$ have finite orders m, n, then there is a $c \in K$ whose order is the least common multiple of m, n. [Hint: First do the case of m, n relatively prime.]

b) Show that if G is finite then it is cyclic. [Hint: Let ℓ be the l.c.m. of the orders of the elements of G, and apply problem 3(b) to the polynomial $x^\ell - 1$.]

c) Conclude that if $K \subset L$ is an extension of finite fields, then $L = K[a]$ for some $a \in K$. [Hint: What is the group structure of L^*?]

The remaining problems are optional, and preserve the notation of problem 2 above.

6. Show that the following conditions are equivalent:

(i) (R, ρ) is a complete metric space.

(ii) (K, ρ) is a complete metric space.

(iii) R is a complete local ring, i.e. $R = \varprojlim R/\mathfrak{m}^n$.

7. Is K compact if $R = \mathbb{F}_p[[x]]$? If $R = \mathbb{F}_p[x]_{(x)}$? If $R = \mathbb{Q}[[x]]$? If $R = \mathbb{Z}_p$ (the p-adic integers)? If $R = \mathbb{Z}_{(p)}$?

8. a) Show that if $a_1, a_2, a_3, \ldots \in K$ and if $\sum_{n=1}^{\infty} a_n$ converges to an element of K, then $\lim_{n\to\infty} a_n = 0$.

b) For which of the rings in problem 7 does the converse to part (a) hold? Can you state and prove a necessary and sufficient condition on R for the converse to hold? Compare and contrast this to the situation for the fields $\mathbb{R}$ and $\mathbb{C}$ under their usual topologies.

9. a) Show that if $f \in K[x]$, then the function $K \to K$ given by f is identically 0 if and only if f is the zero polynomial. Is this true for fields in general?

b) If $f : K \to K$ is a function, define its *derivative* $f' : K \to K$ by the usual expression $f'(a) = \lim_{h\to 0} \big(f(a+h) - f(a)\big)/h$, if this exists for all $a \in K$. Show that if f is given by a polynomial in $K[x]$ then its derivative exists, and compute it. Also, find all polynomial functions f such that f' is the zero function. (Your answer should depend on K.)