Math 624 Problem Set #1 Due Wednesday, Sept. 23, 2020

1. Do Hartshorne, Chapter 1, problems #1.1, 1.2, 1.3, 1.4.

2. Which of the following are real algebraic sets (i.e. algebraic sets over $\mathbb{R}$)? For those that are: Which are connected in the metric topology? Which are connected in the Zariski topology? Which are irreducible (in the Zariski topology)? What are their dimensions? At which points are they smooth (i.e. manifolds)?

(a) $\{(x, y) \in \mathbb{R}^2 \mid y = \sin x\}$
(b) $\{(x, y) \in \mathbb{R}^2 \mid (\exists t)\, x = \cos t, y = \sin t\}$
(c) $\{(x, y) \in \mathbb{R}^2 \mid x + y \in \mathbb{Z}\}$
(d) $\{(x, y) \in \mathbb{R}^2 \mid x + y \in \mathbb{Z}, 0 \leq x + y \leq 10\}$
(e) $\{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$

3. Do the same over the complex numbers, for the loci of:

(a) $x^2 + y^2 = 0$ in $\mathbb{C}^2$
(b) $y^2 = x^2(x - 1)$ in $\mathbb{C}^2$
(c) $y^2 = x(x^2 - 1)$ in $\mathbb{C}^2$
(d) $(x + y)^2 = 0$ in $\mathbb{C}^2$
(e) $xy = 1$ in $\mathbb{C}^2$
(f) $|x| + |y| = 1$ in $\mathbb{C}^2$

4. Let k be a field, let $X \subset \mathbb{A}^n_k$ be a closed subset, and let $I = I(X) \subset A := k[x_1, \ldots, x_n]$ be the ideal of polynomials vanishing identically on X. Let:

$A(X) = \{\text{polynomial functions on } X\} = A/I$, and let

$A'(X) = \{$functions on X expressible in the form f/g, where $f, g \in A$, and where g never vanishes on $X\}$.

(a) Show that $A(X) \subset A'(X)$.

(b) Show that if k is algebraically closed (e.g. $k = \mathbb{C}$) then $A(X) = A'(X)$. [Hint: First use the Weak Nullstellensatz to show that I, g together generate the unit ideal of A.]

(c) Show that $A(X)$ is not necessarily the same as $A'(X)$ if k is not algebraically closed. [Hint: Find a counterexample with $k = \mathbb{R}$, $n = 1$.]