Math 624 Problem Set #2 Due Wednesday, Sept. 30, 2020

Read Hartshorne, Chapter I, sections 1-3.

1. In Hartshorne, Chapter I, do these problems:
   2.5, 2.10, 2.14, 2.15, 3.1, 3.2.

2. (a) Let n be a positive integer, and let $X \subset \mathbb{A}^2_{\mathbb{C}}$ be the curve given by $x^n + y^n = 1$. Show that $\mathbb{A}^1_{\mathbb{C}}$ is birationally isomorphic to X if and only if there exist relatively prime non-zero polynomials $p, q, r \in \mathbb{C}[t]$ such that $p^n + q^n - r^n = 0$. [Hint: Is there an isomorphism of function fields?]

   (b) What happens for $n = 1$ and $n = 2$?

3. In the context of problems 1.11 and 2.17(c) of Hartshorne, Chapter I, find two irreducible polynomials $f, g \in k[x, y, z]$ such that the zero locus $Z = Z(I)$ of the ideal $I := (f, g)$ is a curve in $\mathbb{A}^3$ that contains Y. Check whether $Z = Y$ and whether the ideal (f, g) is equal to $I(Y)$, for your choice of f, g. (If you're feeling ambitious, you could also try to do those starred problems in Hartshorne.)

4. Following Bourbaki, call a topological space *quasi-compact* if every open cover has a finite subcover. Call it *compact* if it is quasi-compact and Hausdorff.

   (a) Show that *every* affine variety is quasi-compact in the Zariski topology, but that *no* affine variety, except for a finite set of points, is compact in the Zariski topology.

   (b) Which affine varieties over $\mathbb{C}$ are compact in the (classical) metric topology?

   (c) Does your answer to (b) remain true over $\mathbb{R}$?