Math 624 Problem Set #3 Due Friday, Oct. 9, 2020

Read Hartshorne, Chapter I, sections 4-5.

1. In Hartshorne, Chapter I, do these problems:
   3.5, 3.6, 3.14(a), 3.21, 4.4, 4.6, 5.1, 5.2.

2. (a) Show that the complex affine line is not birationally isomorphic to the curve $X \subset \mathbb{A}^2_{\mathbb{C}}$ given by $x^3 + y^3 = 1$. Do this in steps, by assuming instead that there *is* an isomorphism $\Phi : K(\mathbb{A}^2) \to K(X)$ of function fields, and then proceeding as follows:

(i) There are relatively prime non-zero polynomials $p, q, r \in \mathbb{C}[t]$ such that $p^3 + q^3 - r^3 = 0$ identically. [Hint: See problem 2 in Problem Set 2.]

(ii) For these p, q, r, there is the matrix identity

$$\begin{pmatrix} p & q & r \\ p' & q' & r' \end{pmatrix} \begin{pmatrix} p^2 \\ q^2 \\ -r^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where p', q', r' are the derivatives of p, q, r.

(iii) Hence $qr' - rq'$, $rp' - pr'$, $pq' - qp'$ are respectively divisible by p^2, q^2, r^2.

(iv) By examining the degrees of p, q, r, derive a contradiction.

(b) What if the base field is some other algebraically closed field k, instead of $\mathbb{C}$? Does the conclusion of part (a) remain true for *all* such k?

3. Let $X = \mathbb{A}^2$, let $P \in X$ be the origin, and let $\pi : \tilde{X} \to X$ be the blow-up of X at P. Let ϕ be the rational map from X to $\mathbb{P}^1$ that sends (x, y) to $(x : y)$. Show explicitly that ϕ lifts to a morphism $\tilde{\phi}$ defined on $\tilde{X}$; i.e., there exists a morphism $\tilde{\phi} : \tilde{X} \to \mathbb{P}^1$ that defines the same rational map as $\phi \circ \pi$.

4. (a) Find a blow-up, or a sequence of blow-ups, that makes the curve $y^2 = x^5$ smooth. Describe the proper transform and the total transform.

(b) Do the same for $y^3 = x^5$.