Math 624 Problem Set #4 Due Monday, Oct. 19, 2020

In Hartshorne, read Chapter I, sections 6-8; and Chapter II, section 1.

1. (a) In Hartshorne, Chapter I, do these problems: 5.3, 5.4, 6.6, 7.1.
   (b) In Hartshorne, Chapter II, do these problems: 1.8, 1.14.

2. Let $P = (0,0) \in \mathbb{A}^2$, $L = (y = 0) \subset \mathbb{A}^2$ over $\mathbb{C}$. Consider the following varieties in $\mathbb{A}^2$:
(i) $y = x^2$; (ii) $y^2 = x^3 + x^2$; (iii) $y^2 = x^3$.

For each of these varieties V, do the following:

(a) Draw the (real points of the) graph of V, and find the singular locus of V.

(b) Find the tangent cone to V at P. Also find the tangent lines to V at P and their multiplicities. [If $I(V) = (f)$ and the initial form (of lowest degree terms) of f is $\prod \ell_i^{n_i}$ where the ℓ_i are linear polynomials defining distinct lines L_i, then the L_i are the *tangent lines* at the origin and the *multiplicity* of L_i is n_i. The *tangent cone* is the union of the tangent lines, counting them with their multiplicities.]

(c) Find the tangent space to V at P in $\mathbb{A}^2$ (i.e. the space defined by the linear terms of the above polynomial f). Also calculate $\dim_{\mathbb{C}}(\mathfrak{m}_P/\mathfrak{m}_P^2)$ directly, where $\mathfrak{m}_P$ is the maximal ideal associated to P. Compare. (Also compare to the space spanned by the tangent lines.)

(d) Find the intersection multiplicity $(L \cdot V)_P$ of L with V at P, and find $(L \cdot V)$. [Cf. Hartshorne, Exercise I 5.4.] Does this agree with the real picture?

3. (a) Show that $\mathrm{GL}_n(k) := \{\text{invertible } n \times n \text{ matrices over } k\}$ and $\mathrm{SL}_n(k) := \{\text{matrices in } \mathrm{GL}_n(k) \text{ of determinant } 1\}$ are *group varieties* over k (cf. Hartshone Chapter I, problem 3.21). Are they affine? If so, in which affine space $\mathbb{A}_k^N$ do they respectively lie as Zariski closed subsets? Is there a natural homomorphism of group varieties between them (i.e. a morphism that is also a group homomorphism)?

(b) What about the orthogonal group $\mathrm{O}_n(k) := \{A \in \mathrm{GL}_n(k) \mid AA^{\mathrm{t}} = I\}$, and the special orthogonal group $\mathrm{SO}_n(k) := \mathrm{O}_n(k) \cap \mathrm{SL}_n(k)$?

(c) What about the unitary group, in the case that $k = \mathbb{C}$?

[Note: Group varieties are also called "algebraic groups". Algebraic groups that are affine are called "linear algebraic groups". Over the field of complex numbers, algebraic groups are also Lie groups, and in particular $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{SL}_n(\mathbb{C})$ are Lie groups.]

4. (a) Let $X = \mathbb{P}^1_{\mathbb{C}}$, regarded as a complex analytic space, i.e. with the metric topology and equipped with the structure sheaf $\mathcal{H}$ of holomorphic functions. Let n be a positive integer. Consider the morphism of sheaves $\phi : \mathcal{H}^\times \to \mathcal{H}^\times$ given on each open set by $f \mapsto f^n$. For which open sets $U \subset X$ is $\phi(U) : \mathcal{H}^\times(U) \to \mathcal{H}^\times(U)$ injective? surjective? Find the sheaf kernel and sheaf cokernel of ϕ. Into what short exact sequence does this morphism fit?

(b) What changes if X is instead regarded as an algebraic variety, with the Zariski topology and structure sheaf $\mathcal{O}$, and taking $\phi : \mathcal{O}^\times \to \mathcal{O}^\times$? (Remark: It is possible to recapture the situation in part (a) for algebraic varieties if instead one uses the *étale* topology.)