Math 624 Problem Set #5 Due Friday, Oct. 30, 2020

In Hartshorne, read Chapter II, sections 2-3.

1. In Hartshorne, Chapter II, do these problems: 1.17, 2.8, 2.10, 2.11, 2.13, 3.5, 3.9, 3.16.

2. a) Let k be a field, and draw $\operatorname{Spec} k[x,t]$, $\operatorname{Spec} k[x,t]/(t^2)$, and $\operatorname{Spec} k[x][[t]]$. In each case draw the following loci, showing and labeling each point where they meet:
$(x), (x-2), (t), (t-1), (t-x), (t-x^2), (t^2-x), (1-xt)$.
Indicate if your answers depend on the field, and if so how.

b) For each of the following rings R, sketch $X = \operatorname{Spec} R$ and determine whether X is connected, irreducible, reduced, integral: $\mathbb{C}[x]/(x^2-x)$, $\mathbb{C}[x]/(x^3-x^2)$, $\mathbb{C}[x,y]/(xy-y)$, $\mathbb{Z}/6$, $\mathbb{Z}/12$, $\mathbb{Z}[x]/(2x)$, $\mathbb{Z}[x]/(x^2)$, $\mathbb{Z}[x]/(4)$.

3. Let R be a ring and let $X = \operatorname{Spec} R$.

a) Show that the nilradical of R (i.e. the intersection of the prime ideals of R) is trivial iff X is reduced.

b) Suppose that the nilradical of R is trivial. Show that the Jacobson radical of R (i.e. the intersection of the maximal ideals of R) is trivial iff $\overline{\operatorname{Max} R} = X$ (where we view $\operatorname{Max} R \subseteq \operatorname{Spec} R = X$, and take its closure).

c) Give an example of a ring such that either the nilradical or the Jacobson radical is trivial, but not both. For this example, verify directly that the assertions of a) and b) hold.

4. Let k be a field. For each of the following morphisms ϕ of schemes, determine whether ϕ is of finite type, finite, quasi-finite, or surjective.
(i) ϕ is the morphism corresponding to the endomorphism of $k[x]$ given by $x \mapsto x^3$.
(ii) ϕ is the morphism corresponding to the inclusion of rings $k[x] \hookrightarrow k[x,y]/(y^3-y-x)$.
(iii) ϕ is the morphism corresponding to $k[x] \hookrightarrow k[x,y,y^{-1}]/(y^3-y-x)$.
(iv) ϕ is the first projection map $\mathbb{P}^1_k \times_k \mathbb{A}^1_k \to \mathbb{P}^1_k$.
(v) ϕ is the second projection map $\mathbb{P}^1_k \times_k \mathbb{A}^1_k \to \mathbb{A}^1_k$.
(vi) ϕ is the morphism corresponding to the inclusion of rings $k[x] \hookrightarrow k(x)$.