In Hartshorne, read Chapter II, sections 6-7.

1. In Hartshorne, Chapter II, do these problems: 6.6, 6.7, 7.1, 7.5, 7.6. (Optional study problems to read, not to be submitted: 6.10-6.12, 7.7, 7.13.)

2. Let k be an algebraically closed field, and let X be a smooth connected projective k-curve that is not isomorphic to $\mathbb{P}^1_k$. Let K be the function field of X. Let $f \in K - k$.

a) Show that f defines a non-constant rational map from X to $\mathbb{P}^1$, and that this extends to a morphism $X \to \mathbb{P}^1$.

b) Deduce that the divisor $(f)_\infty$ has degree > 1. [Hint: What is the degree of the morphism in (a), or equivalently the degree of the corresponding field extension?]

c) Deduce that if P is a closed point of X, then there is no rational function on X having a pole of order 1 at P and having no other poles.

d) Conclude that if $P, Q \in X$ are distinct closed points, then viewed as divisors, P and Q are *not* linearly equivalent.

e) Evaluate the dimensions of the k-vector spaces $\Gamma(X, \mathcal{O})$ and $\Gamma(X, \mathcal{O}(P))$, where P is a closed point of X.

f) Do your answers to parts (a)-(e) change if we instead take $X = \mathbb{P}^1$?

3. a) Let Y_1, Y_2 be distinct irreducible curves in $\mathbb{P}^2$ of degrees d_1, d_2 respectively. Let U be the complement of $Y_1 \cup Y_2$ in $\mathbb{P}^2$. Find the divisor class group of U, and determine whether it is trivial and whether it is torsion free.

b) Consider $\mathbb{P}^1 \times \mathbb{P}^1$, with coordinates $(x_0 : x_1; y_0 : y_1)$. Let d be a positive integer, and let Y be the curve in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equation $x_1^d y_0^d + x_0^d y_1^d = x_0^d y_0^d$. Let U be the complement of Y in $\mathbb{P}^1 \times \mathbb{P}^1$. Find the divisor class group of U and determine whether it is trivial and whether it is torsion free.

4. a) Show that the quartic (degree 4) curves in $\mathbb{P}^2$ form a complete linear system, and find its dimension d. (Here "curve" means the scheme defined by the ideal of a homogeneous polynomial, and degenerate curves are permitted.)

b) Let P be a closed point of $\mathbb{P}^2$, and consider the curves in the linear system in (a) that pass through P. Show that they form a linear system, and find its dimension. Is this a complete linear system?

c) Redo part (b) with P replaced by two distinct points P, Q in $\mathbb{P}^2$ (i.e. curves passing through both points).

d) Does the obvious pattern of dimensions, suggested by your answers to parts (a)-(c), continue indefinitely if more and more points are chosen?