Math 624 Problem Set #9 Due Thursday, Dec. 10, 2020

Note: Those who would like extra time can have an automatic extension of up to four days.

In Hartshorne, read Chapter II, sections 8-9.

1. In Hartshorne, Chapter II, do these problems: 7.10(a,b) (parts (c,d) are optional); 8.5(a) (part (b) is optional); 8.8, 9.1.

2. a) Show directly, by considering differential forms, that $\Omega^1_X \approx \mathcal{O}(-2)$, if $X = \mathbb{P}^1$. [Hint: What is $d(x^{-1})$?]

b) Show directly, by considering differential forms, that $\Omega^1_X \approx \mathcal{O}$, if $X \subset \mathbb{P}^2$ is the cubic curve given by $y^2z = x^3 - xz^2$.

c) Verify Riemann-Roch directly for $X = \mathbb{P}^1$. That is, show that for any divisor D,

$$\dim \Gamma(X, \mathcal{O}(D)) - \dim \Gamma(X, \Omega^1_X \otimes \mathcal{O}(-D)) = \deg D + 1 - g,$$

where $g = \text{genus}(\mathbb{P}^1) = 0$. [Hint: $D \sim nP$ for some $n \in \mathbb{Z}$ and any point P on $\mathbb{P}^1$.]

3. a) Show that for every integer $n > 1$ there is an integer $d > 1$ with the following property: if m is any positive integer and $\phi : \mathbb{P}^m \to \mathbb{P}^n$ is a morphism, then the image of ϕ cannot be a smooth hypersurface of degree at least d. Find such a d explicitly in terms of n. [Hint: ω.]

b) Find a pair of integers $n, d > 1$ such that the above property fails for that pair (i.e., d is not sufficiently large with respect to n).

4. Let k be a field, let T be the affine t-line over k, let $X = \text{Spec } k[x, y, t]/(y^2 - x^3 - t)$, and let $f : X \to T$ be the projection onto the t-coordinate. Determine where X is regular, where it is smooth as a k-scheme, and where it is smooth as a T-scheme. (Note: Your answer should depend on the characteristic of k.)