Math 624 | Problem Set #10 | Due Monday, Dec. 9, 2002

Part I:

Read Fulton, Chapter 6, sections 4-6.
Do problems:

5.38 on pp.127;
6.27, 6.29, 6.30 on pp.147-148;
6.38, 6.39, 6.45 on pp.156-157.

Part II:

Read Hartshorne, Chapter II, sections 6-8. (Section 9 is optional.)

1. In Chapter II, do problems 6.6, 7.1.
(Optional: In Chapter II, do problems 6.1, 6.7, 6.12, 7.3-7.5, 7.9, 8.8.)
(Additional study problems: 6.2, 6.4, 6.10, 7.6, 7.7, 7.13, 8.2, 8.3.)

2. a) Show that the quartic (degree 4) curves in $\mathbb{P}^2$ form a complete linear system, and find its dimension d. (Here "curve" means the scheme defined by the ideal of a homogeneous polynomial, and degenerate curves are permitted.)

b) Let P be a closed point of $\mathbb{P}^2$, and consider the curves in the linear system in (a) that pass through P. Show that they form a linear system, and find its dimension. Is this a complete linear system?

c) Redo part (b) with P replaced by two distinct points P, Q in $\mathbb{P}^2$ (i.e. curves passing through both points).

d) Does the obvious pattern of dimensions, suggested by your answers to parts (a)-(c), continue indefinitely if more and more points are chosen?

3. Let k be an algebraically closed field, and let X be a smooth connected projective curve that is not isomorphic to $\mathbb{P}^1_k$. Let K be the function field of X. Let $f \in K - k$.

a) Show that f defines a non-constant rational map from X to $\mathbb{P}^1$, and that this extends to a morphism $X \to \mathbb{P}^1$.

b) Deduce that the divisor $(f)_\infty$ has degree > 1. [Hint: What is the degree of the morphism in (a)?]

c) Deduce that if P is a closed point of X, then there is no rational function on X having a pole of order 1 at P and having no other poles.

d) Conclude that if $P, Q \in X$ are distinct closed points, then viewed as divisors, P and Q are *not* linearly equivalent.

e) Evaluate the dimensions of the k-vector spaces $\Gamma(X, \mathcal{O})$ and $\Gamma(X, \mathcal{O}(P))$, where P is a closed point of X.

f) Do your answers to parts (a)-(e) change if we instead take $X = \mathbb{P}^1$?

4. a) Show directly, by considering differential forms, that $\Omega^1_X \approx \mathcal{O}(-2)$ if $X = \mathbb{P}^1$.

b) Show directly, by considering differential forms, that $\Omega^1_X \approx \mathcal{O}$, if $X \subset \mathbb{P}^2$ is the cubic curve given by $y^2 z = x^3 - xz^2$.

c) Verify Riemann-Roch directly for $X = \mathbb{P}^1$. That is, show that for any divisor D,

$$\dim \Gamma(X, \mathcal{O}(D)) - \dim \Gamma(X, \Omega^1_X \otimes \mathcal{O}(-D)) = \deg D + 1 - g,$$

where $g = \text{genus}(\mathbb{P}^1) = 0$. [Hint: $D \sim nP$ for some $n \in \mathbb{Z}$ and any point P on $\mathbb{P}^1$.]