Math 625 | Problem Set #4 | Due Wednesday, March 5, 2003

Read Hartshorne, Chapter IV, section 5.
Read Fulton, Chapter 8, sections 4-5.

1. In Hartshorne, Chapter IV, do problems 3.2, 4.9. Optional: problems 2.5, 3.3, 4.6.

2. In Fulton, Chapter 8, do problems 8.2 (p.191); 8.27 and 8.28 (p.209).

3. Let X be a smooth connected projective curve of genus $g \geq 2$ over an algebraically closed field k. Let $X^{(g)}$ be the gth-fold symmetric power of X. Identify effective divisors D of degree g on X with points of $X^{(g)}$.

a) Show that for all D in some dense open subset of $X^{(g)}$, the complete linear system $|D|$ has dimension 0. [Hint: What is the dimension of $|K|$? What is the dimension of $|K - P_1|$, if P_1 is not a base point of $|K|$? ... What about the dimension of $|K - D|$?]

b) Deduce that $X^{(g)} \to \mathrm{Pic}^0(X)$ is injective on a Zariski open dense subset. (Here the map is given by $D \mapsto [D - D_0]$ for some fixed effective divisor D_0 on X of degree g.)

4. Fix $X = \mathbb{P}^2$ over an algebraically closed base field k. Show that there is a fine moduli space for the lines in X, and that this moduli space is isomorphic to $\mathbb{P}^2$. (First, you have to describe the corresponding functor that you will represent.)

5. Let X be a smooth connected projective curve of genus $g > 1$, and let $n \geq 3$. Show that the n-canonical map associated to the linear system $|nK|$ is an embedding.

6. Let $Y \to X$ be a Galois branched cover of regular connected schemes of dimension 1, with Galois group G. Let $Q \in Y$.

a) Show that if $g \in G$ is in the decomposition group D_Q (see PS3, problem 6), then g induces an automorphism on the complete local ring $\hat{\mathcal{O}}_{Y,Q}$ and hence on the residue field.

b) Define the *inertia group* I_Q to consist of the elements $g \in D_Q$ such that g induces the identity on the residue field at Q. Show that I_Q is a subgroup of D_Q and of G. Must it be a *normal* subgroup of D_Q? of G?

c) Find the inertia groups I_Q for each of the examples in PS3 problem 6(c)-(e).

d) Let k be the field $\mathbb{F}_2(u)$, let X be the affine x-line over k, let Y be the curve in the x, y-plane over k given by $y^2 - xy = u$, and let $Y \to X$ be the projection onto the x-coordinate. Show that the cover is Galois; find the Galois group; and find the branch locus. Also find D_Q, I_Q, and e_Q if Q lies over $x = 0$.

e) Show that the order of I_Q is equal to the ramification index e_Q, if the residue field k is perfect. Does this conclusion hold without any assumption on k? Compare this with the assertion of PS3 problem 6(b) and with the examples in parts (c) and (d) above.