Math 625 Problem Set #8 Due Monday, April 21, 2003

Read Hartshorne, Chapter III, sections 11-12; Chapter V, section 1; Appendix A, sections 1-2; and Appendix C, section 3. (Optional: Read the rest of Appendix C.)

1. In Hartshorne, Chapter III, do problems 4.5 (for a Noetherian separated scheme), 9.4, 10.2, 11.2, 11.3.
Optional: problems 9.11, 10.1, 10.3, 10.5, 10.6, 11.4, 12.2.

2. For each of the following morphisms ϕ, determine whether ϕ is of finite type, finite, quasi-finite, proper, surjective, projective, flat, and étale. (Below, k is an arbitrary field.)
(i) ϕ is the morphism corresponding to the inclusion of rings $k[x] \hookrightarrow k[x,y]/(y^2 - x^2)$.
(ii) ϕ is the morphism corresponding to the inclusion of rings $k[x] \hookrightarrow k[[x]]$.
(iii) ϕ is the morphism corresponding to the inclusion of rings $k[x] \hookrightarrow k[x,y]/(y^3 - x)$.
(iv) ϕ is the morphism corresponding to the inclusion of rings $k[x] \hookrightarrow k[x,x^{-1},y]/(y^3 - x)$.
(v) ϕ is the morphism corresponding to the inclusion of rings $k[x] \hookrightarrow k[x,y]/(xy)$.
(vi) ϕ is the morphism corresponding to the inclusion of rings $k[x,y] \hookrightarrow k[x,y,z]/(z^2 - xy)$.
(vii) ϕ is the morphism corresponding to the inclusion of rings $k[x,y,z]/(z^2 - xy) \hookrightarrow k[u,v]$ given by $x \mapsto u^2$, $y \mapsto v^2$, $z \mapsto uv$.
(viii) $\phi : E \to E$ is multiplication by 3 on an elliptic curve over k.

3. Let k be a field. Which of the following ideals $I \subset k[[x,y]]$ is maximal? prime? the unit ideal? In each case, describe geometrically the locus of I in $\operatorname{Spec} k[[x,y]]$.
$I = (x), (x,y), (xy), (1 - xy), (x - y), (y^2 - x^2), (y^2 - x^3), (y^2 - x^2 - x^3)$.

4. Let $f : Y \to X$ be a finite étale cover of smooth connected schemes, say of degree n. Show that there is a finite étale cover $Z \to X$ such that the pullback $Y \times_X Z \to Z$ is a trivial cover, consisting of n disjoint copies of Z. Explain why this says that finite étale covers are covering spaces in the étale topology. Contrast this with what happens in the Zariski topology.

5. Suppose that $f : Y \to X$ is a birational morphism of smooth projective varieties, and let $H \subset Y$ be a hypersurface whose image has dimension less than that of H. Prove that H is not linearly equivalent to any effective divisor on Y that meets H properly. [Hint: Otherwise, consider the corresponding rational function on Y, and view it as a rational function on X. What is its divisor there?]