

Read Hartshorne, Chapter V, sections 2-6. (Optional: Read Appendix A, sections 3-5; and Appendix B.)

1. In Hartshorne, Chapter V, do problems 1.5, 2.1, 4.5.

Optional: problems 1.7, 1.12, 2.6, 3.1.

2. What does Riemann-Roch say about the dimension of the space of rational functions on \mathbb{P}^2 which have poles at worst D , where D is a given effective divisor of degree d ? How can this conclusion also be seen without Riemann-Roch?

3. a) Find an example of a smooth projective surface X , an ample divisor H on X , and a divisor D on X , such that $D \cdot H > 0$ but nD is not linearly equivalent to an effective divisor for any positive integer n .

b) In your example, what is D^2 ?

c) Can there be an example in part (a) for which $D^2 > 0$?

4. Let X be the blow-up of \mathbb{P}^2 at a point P .

a) Describe $\text{Pic } X$ as a group together with the intersection pairing. Determine which divisors on X are ample.

b) Determine which divisors on X are ample.

c) Verify the Hodge Index Theorem for X .

5. Consider the rational ruled surface X , mapping onto the projective x -line \mathbb{P}^1 , which is constructed as follows. Over the affine x -patch U_0 of \mathbb{P}^1 , the inverse image is $U_0 \times \mathbb{P}^1$, where the second factor is the projective y -line. Over the affine \bar{x} -patch U_1 of \mathbb{P}^1 (where $x\bar{x} = 1$ on $U_{01} := U_0 \cap U_1$), the inverse image is the projective \bar{y} -line. Over U_{01} , we have the transition function $\bar{y} = \bar{x}^n y$, for some non-negative integer n . Find the numerical invariant e of the rational ruled surface X , and find a locally free sheaf \mathcal{E} of rank 2 on \mathbb{P}^1 such that $X \approx \mathbb{P}(\mathcal{E})$.

6. a) Let H be the line at infinity in \mathbb{P}^2 , and let P, Q be distinct points on H . Let X be the blow-up of \mathbb{P}^2 at P and Q ; let E_1, E_2 be the exceptional divisors over P, Q ; and let L be the proper transform of H . Describe $\text{Pic } X$, in particular giving the self-intersections of E_1, E_2, L , and giving the pairwise intersections of these three divisors.

b) Let H_1, H_2 be the two lines at infinity in $\mathbb{P}^1 \times \mathbb{P}^1$, given by $x = \infty$ and $y = \infty$ respectively. Let O be the point at which H_1, H_2 intersect. Let X' be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at O ; let E be the exceptional divisor over O ; and let L_1, L_2 be the proper transforms of H_1, H_2 . Describe $\text{Pic } X'$, in particular giving the self-intersections of L_1, L_2, E , and giving the pairwise intersections of these three divisors.

c) Show that $X \approx X'$. Under your isomorphism, which divisors of X' do $E_1, E_2, L \subset X$ correspond to?