Math 625 Problem Set #7 Due Monday, April 19, 2021

In Hartshorne, read Chapter III, Sections 10-12; and Chapter V, Section 1. Optional: Read Appendices B and C.

1. From Hartshorne: In Chapter III, do problems 10.1, 10.6. In Chapter V, do problems 1.4, 1.5. Optional: In Chapter III do problems 10.3, 10.5, 11.4, 12.2, and in Chapter V, do problems 1.7, 1.12.

2. (a) Show that the symmetric power $\mathrm{Sym}^n(\mathbb{P}^1)$ is isomorphic to $\mathbb{P}^n$, over any algebraically closed field. [Hint: View a point of $\mathbb{P}^n$ as corresponding to the coefficients of a polynomial $f(x)$, and a point of $\mathrm{Sym}^n(\mathbb{P}^1)$ as corresponding to the roots of the polynomial.] What happens over $\mathbb{R}$?

(b) Let $\Delta \subset \mathrm{Sym}^n(\mathbb{P}^1)$ be the "weak diagonal," consisting of elements with at least two equal entries. Explain why this corresponds to the "discriminant locus" of $\mathbb{P}^n$ (meaning the locus where the discriminant of the corresponding polynomial vanishes).

3. (a) Show that if E/F is a Galois field extension of degree d, then $E \otimes_F E$ is isomorphic to a direct sum of d copies of E.

(b) Deduce that if $Y \to X$ is a Galois finite étale cover of integral schemes of degree d, then $W := Y \times_X Y$ is isomorphic to a disjoint union of d copies of Y.

(c) Deduce that if $Y \to X$ is a finite étale cover of integral schemes of degree d, and if $Z \to X$ is the Galois closure of $Y \to X$, then $Z \times_X Y$ is isomorphic to a disjoint union of d copies of Z.

4. Let X be a smooth connected projective variety, let P be a closed point of X, and let $\tilde{X}$ be the blow-up of X at P, with exceptional divisor E. Show that E is not linearly equivalent in $\tilde{X}$ to any effective divisor on $\tilde{X}$ whose support does not contain E. [Hint: Otherwise, consider the corresponding rational function on Y, and view it as a rational function on X. What is its divisor there?]

5. What does Riemann-Roch say about the dimension of the space of rational functions on $\mathbb{P}^2$ that have poles at worst D, where D is a given effective divisor of degree d? How can this conclusion also be seen without Riemann-Roch?