Topics in Algebra – Math 702

Problem Set #2 Due Wed., March 16, 2020, in class.

1. Let k be a field of characteristic unequal to 2. Let $F = k((t))$ and $R = k[[t]]$.

a) Let f be an element of R with constant term c. Show that f is a unit in the ring R if and only if $c \neq 0$. Also show that if $c = 1$ then f is a square in R. [Hint: Taylor series for $(1+x)^{1/2}$.] Deduce that if $c \neq 0$, then f is a square in R (and in F) if and only if $c \in k^{\times 2}$.

b) Let $q = \langle a_1, \ldots, a_n \rangle$ with $a_i \in R^\times$, the group of units in R. Show that if q is isotropic over F then $q(x) = 0$ for some $x = (x_1, \ldots, x_n) \in R^n$ that does not lie in tR^n.

c) In (b), write $a_i = c_{i,0} + c_{i,1}t + c_{i,2}t^2 + \cdots$ with $c_{i,j} \in k$, and write $\bar{q} = \langle c_{1,0}, \ldots, c_{n,0} \rangle$. Show that if q is isotropic over F then $\bar{q}$ is isotropic over k. [Hint: Use part (b) and then reduce mod (t).]

d) Prove the converse of part (c). [Hint: Use part (a).]

2. In this problem, we retain the notation of problem 1.

a) Show that every regular quadratic form over F is equivalent to a quadratic form $q_1 \perp tq_2$ for some $q_1 = \langle a_1, \ldots, a_r \rangle$ and some $q_2 = \langle a_{r+1}, \ldots, a_n \rangle$, where each $a_i \in R^\times$. Show moreover that if $\bar{q}_1$ or $\bar{q}_2$ is isotropic, then so is q. [Hint: Use problem 1(d).]

b) Prove the converse of the last part of (a). [Hint: First obtain an $x \in R^n$ as in problem 1(b). Next, consider the case in which at least one of the elements $x_1, \ldots, x_r \in R$ has non-zero constant term; and handle this case by modding out by (t). Finally, handle the remaining case by showing that the form $t^2 q_1 + tq_2$ is also isotropic over R, and then dividing by t and reducing to the previous case.]

c) Using parts (a) and (b), find and prove a formula that relates $u(F)$ to $u(k)$.

3. Let $\mathbb{H}$ be the usual (Hamiltonian) quaternion algebra over $\mathbb{R}$.

a) Show by example that a polynomial of degree n over $\mathbb{H}$ can have more than n roots in $\mathbb{H}$.

b) Explain where the usual proof that this cannot happen in a field breaks down in the division algebra $\mathbb{H}$.

c) Explain why a factorization $f(X) = g(X)h(X)$ of polynomials over $\mathbb{H}$ does not in general imply that $f(c) = g(c)h(c)$ for $c \in \mathbb{H}$, though it does if the coefficients of f, g, h lie in $\mathbb{R}$. [Note: this is related to part (b).]

4. Let $f(X) \in \mathbb{R}[X]$.

a) Show that if $\alpha \in \mathbb{H}$ is a root of $f(X)$, then so is $\beta\alpha\beta^{-1}$ for all $\beta \in \mathbb{H}^\times$.

b) Find all the square roots of -1 in $\mathbb{H}$, and show that this is consistent with part (a).

5. Let $a \in \mathbb{H}$.

a) Write $f(X) = X^2 - a$, $\bar{f}(X) = X^2 - \bar{a}$, and $F(X) = \bar{f}(X)f(X)$. Show that $F(X) \in \mathbb{R}[X]$, and that $F(X)$ has a root α in $\mathbb{C} = \mathbb{R}[i] \subset \mathbb{H}$.

b) Show by direct computation that if $c := f(\alpha) \neq 0$ then $\beta := \overline{c\alpha c^{-1}}$ is a root of $f(X)$.

c) Conclude that a has a square root in $\mathbb{H}$.

[Note: This argument can be generalized to show that $\mathbb{H}$ is "algebraically closed" as a division algebra.]

6. Let $a \in \mathbb{H}$ such that $a \notin \mathbb{R}$.

a) Show that $K := \mathbb{R}(a) \subset \mathbb{H}$ is a degree two field extension of $\mathbb{R}$; that K is a maximal subfield of $\mathbb{H}$; and that the centralizer $C_{\mathbb{H}}(K)$ of K in $\mathbb{H}$ is equal to K.

b) Show that a has *exactly* two square roots in $\mathbb{H}$. [Hint: Show that any square root of a must commute with a and must therefore lie in K, which is a field.]

c) Where did you use that $a \notin \mathbb{R}$? What happens if $a \in \mathbb{R}$?