Topics in Algebra – Math 702

Problem Set #5 Due Wed., April 29, 2020.

1. a) Which of the following elements lie in $\mathbb{F}_3((t))$? For each one that does, find this element explicitly, i.e. in the form $\sum_{i=n}^{\infty} a_i t^i$, where $n \in \mathbb{Z}$ and each $a_i \in \mathbb{F}_3$.

$$\frac{1+t^3}{t^2},\ \frac{1}{1-t},\ \sqrt{1+t},\ \sqrt{t},\ \sqrt{2+t}$$

b) Which of the following elements lie in $\mathbb{Q}_3$? For each one that does, find this element explicitly, i.e. in the form $\sum_{i=n}^{\infty} a_i 3^i$, where $n \in \mathbb{Z}$ and each $a_i \in \mathbb{Z}$. If you can, write the element in a form so that each $a_i \in \{0, 1, 2\}$.

$$\frac{41}{9},\ -\frac{1}{2},\ \frac{1}{2},\ \sqrt{7},\ \sqrt{2},\ \sqrt{3}$$

c) Prove that $(\mathbb{Z}_2^\times)^2 = \{a \in \mathbb{Z}_2 \mid a \equiv 1 \pmod 8\}$. [Hint: Strong Hensel's Lemma.]

2. a) Find all anisotropic quadratic forms over $\mathbb{F}_3((t))$ up to equivalence.

b) Which of these forms remain anisotropic over $\mathbb{F}_9((t))$? Which of them become isometric to each other over $\mathbb{F}_9((t))$?

c) Redo part (b) with $\mathbb{F}_3((\sqrt{t}))$ instead of $\mathbb{F}_9((t))$.

3. Consider the following quadratic forms over $\mathbb{Q}$:

$$q_1 = \langle 1, 1, 1, -1 \rangle,\ q_2 = \langle 1, 1, 1, -7 \rangle,\ q_3 = \langle 1, 1, 1, 1, 1 \rangle.$$

a) For each i, determine whether the form q_i is isotropic over $\mathbb{Q}$.

b) For each i, find all the completions of $\mathbb{Q}$ (i.e. $\mathbb{Q}_p$ or $\mathbb{R}$) over which q_i is isotropic. Does your answer, in conjunction with your answer to part (a), agree with Hasse-Minkowski?

4. Let q be a quadratic form over a global field F, and let Ω be the set of (equivalence classes of) non-trivial absolute values on F.

a) Prove that q is hyperbolic over F if and only if it is hyperbolic over F_v for all $v \in \Omega$.

b) Let i_W denote the Witt index of a quadratic form, and for $v \in \Omega$ let q_v denote q viewed as a form over F_v. Prove that $i_W(q) = \min_{v \in \Omega} i_W(q_v)$.

5. Let k be a field and let $K = k((t))$, under the t-adic metric. Let $R_1 = k[[t]][x]$, $R_2 = k[x][[t]]$, $R_3 = k[[x, t]]$, and $R_4 = K[[x]]$.

a) Show that $R_1 \subset R_2 \subset R_3 \subset R_4$, and that each of these inclusions is strict.

b) Regard elements of R_1, R_2, R_3 as elements of R_4, i.e. as power series that can be viewed as "Taylor expansions" about the origin in the x-line over the complete field K. Show that every element of R_1 (but not every element of R_2) converges for all $x \in K$; that every element of R_2 (but not every element of R_3) converges for all x with $|x|_t \le 1$; and that every element of R_3 (but not every element of R_4) converges for all x with $|x|_t < 1$. For which values of x in K do all the elements of R_4 converge?