Review:

Quadratic forms $/F$ ← char $\neq 2$

$$q \longleftrightarrow B \longleftrightarrow M$$

Witt Decomposition

(tot. isotropic, $\underbrace{\text{hyperbolic, anisotropic}}_{\text{regular}}$)

$u(F)$ = max'l dim of anisotropic q.f. $/F$
= min'l dim st every reg. qf$_{/F}$ is universal

Witt Cancellation holds.

Witt index $\quad i_w(q)$ = # of hyp. planes in q

Grothendieck-Witt group $\hat{W}(F)$.

mod hyperbolics: Witt group $W(F)$;
also a ring. $(\perp, \otimes)$

$$\begin{tikzcd}
\hat{I}(F) \arrow[r, phantom, "\subset"] \arrow[d, "\cong"'] & \hat{W}(F) \arrow[r, two heads, "\dim"] \arrow[d, "\mathrm{mod}\ \mathbb{Z}h"] & \mathbb{Z} \arrow[d] \\
I(F) \arrow[r, phantom, "\subset"] & W(F) \arrow[r, two heads, "\dim_0"'] & \mathbb{Z}/2
\end{tikzcd}$$

ideal

$$\begin{tikzcd}
1 \arrow[r] & F^\times/F^{\times 2} \arrow[r] \arrow[dd, phantom, "\wr\wr"] & Q(F) \arrow[r] \arrow[dd, phantom, "\wr\wr"] & \mathbb{Z}/2 \arrow[r] \arrow[d, phantom, "\wr\wr"] & 0 \\
 & & & \{\pm 1\} \arrow[r] \arrow[d, phantom, "\wr\wr"] & 1 \\
1 \arrow[r] & I(F)/I^2(F) \arrow[r] & W(F)/I^2(F) \arrow[r] & W(F)/I(F) \arrow[r] & 1
\end{tikzcd}$$

Quaternion algebras $/F$ ← char $\neq 2$

$A = \left(\frac{a,b}{F}\right)$; $\left(\frac{-1,-1}{\mathbb{R}}\right) = \mathbb{H}$.

Central simple alg $/F$;

division alg or $M_2(F)$. ("split")

↑ iff $\langle a,b\rangle$ reps. 1.

Norm form on A: $\langle 1, -a, -b, ab\rangle$. (q.f.)

Thm/ If A is a csa $/F$ of dim $=4$ then $A = F$ or a quaternion alg $/F$.

Pf uses Wedderburn Structure Thm:

$$A \overset{\text{f.d.}}{\text{csa}}/F \Rightarrow A \cong M_n(D)$$

D div alg $/F$

Quadratic forms	Csa's
monoid under $\perp$	monoid under $\otimes$
Witt equiv: mod hyperbolics	Brauer equiv: mod mx algs
Witt group $W(F)$	Brauer group $Br(F)$
Unique anisotropic qf in each class	Unique div. alg. in each class

For csa A, $\dim A = d^2$, d = degree.

On A, reduced norm is a homog form of deg d in d^2 vbls. (Gen'lizes case of quat. alg.)

Ex. F alg. closed $\Rightarrow Br(F)$ trivial

Ex. $F = \mathbb{R} \Rightarrow Br(F) = \{\pm 1\}$ (Thm of Frobenius)

Ex. F finite $\Rightarrow Br(F)$ trivial

New Ex. k alg cl field, $F = k(x)$
(or finite extension of $k(x)$;
function field of alg. curve $/k$)

Recall: $u(F) = 2$:
Every quad. form. $/F$ in
> 2 vbls is isotropic.

More holds: F satisfies this $\forall d$:
Every homog. poly $/F$ of deg d
in $> d$ vbls is isotropic (*)
(i.e. has a nontrivial sol'n in F).

This fact for F is a thm of Tsen
(proven by Galois cohomology).

In gen'l: Fields with this property (*)
are called quasi-alg. closed (QAC)
or C_1.

Includes alg. cl. flds:
take value of all but
one vbl, solve for other.

For $F = k(x)$ as above (or a fin. ext.),

Let A be a csa of deg $= d > 1$ (so $A \neq F$). So red. norm is homog poly/F of deg d in d^2 vbls.

Tsen's Thm $\Rightarrow$ norm form is isotropic $\Rightarrow$ A is not a div. alg /F.

<u>Conclusion:</u> There are <u>no</u> non-trivial div. alg's /F.

$\therefore$ $Br(F)$ is trivial.

$\uparrow$ sometimes <u>this</u> is called Tsen's Thm.

Re C_1: $\forall n \; \exists$ condition C_n.

$$\underset{\substack{\shortparallel \\ \text{alg. closed}}}{C_0} \Rightarrow C_1 \Rightarrow C_2 \Rightarrow \cdots$$

Recall: if $S \subseteq A$ csa, have centralizer $C_A(S)$.

e.g. $S = B$, subalg of A; have $C_A(B)$.

If B is simple, this has good properties:

Prop A a csa$/F$, $B \subseteq A$ simple subalg, $C := C_A(B)$. Then

i) C is a simple subalg of A.

ii) $\dim A = \dim B \dim C$.

(Lam, Chap IV, Prop 1.6 (1), (3))

Pf of (i) is similar to pf of Wedderburn's Structure Thm. Sketch:

Set $T := \underset{\text{simple}}{B} \otimes \underset{\text{csa}}{A^{op}}$; so T simple.

Take a min'l left T-module S; take $D := \mathrm{End}_T(S)$; get D a div. alg.

A is a left T-module (by $(b \otimes a^{op})a' = b a' a$); this gives

$$\mathrm{End}_T(A) \cong C$$
$$\text{left mult by } c \leftarrow\!\shortmid c$$

Have $A \cong S^m$ as T-module, so then

$$C \cong \mathrm{End}_T A \cong \mathrm{End}_T(S^m) = M_m(\mathrm{End}_T S) = M_m(D); \text{ simple.}$$

For (ii), $T \cong M_n(D^{op})$, where $n = \dim_D S$ (Wedderburn)

$\Rightarrow \dim T = n^2 \dim \overbrace{D}^{= \dim D^{op}}$,

so $(\dim S)^2 = (n \dim D)^2 \underset{(*)}{=} \dim T \cdot \dim D$.

Also $\dim A \underset{(1)}{=} m \dim S$ since $A \cong S^m$.

$\dim T \underset{(2)}{=} \dim A \cdot \dim B$ since $T = B \otimes A^{op}$

$\dim C \underset{(3)}{=} m^2 \dim D$ since $C \cong M_m(D)$

So $\left(\frac{\dim A}{m}\right)^2 \underset{\text{by } (1)}{=} (\dim S)^2 \underset{\text{by } (*)}{=} \dim T \cdot \dim D$

$\underset{\text{by } (2),(3)}{=} \dim A \cdot \dim B \cdot \frac{\dim C}{m^2}$,

So $\dim A = \dim B \dim C$. ✓

<u>Cor</u> (Double Centralizer Thm) (Lang, Ch IV, Prop 1.6 (2))

Let A be a csa $/F$, $B \subseteq A$ simple subalg,

$C := C_A(B)$. Then $B = C_A(C)$.

<u>Pf.</u> Let $B' = C_A(C)$. Then $B \subseteq B'$,

so $\dim B \leq \dim B'$. Applying Prop to

$B \subseteq A$ and to $C \subseteq A$, get:

$\dim B \dim C = \dim A = \dim C \dim B'$,

so ✓.

Cor. If $B \subseteq A$, both csa/F, and $C = C_A(B)$, then $B \otimes C \cong A$.

Pf. B csa, C simple $\Rightarrow$ $B \otimes C$ simple.

$B \otimes C \longrightarrow A$, $b \otimes c \mapsto bc$

is inj (otherwise ker violates simplicity of $B \otimes C$)

Both have same dim (by Prop), so ✓.

Skolen-Noether Thm: (Lam, Ch. IV, Th 1.8)

Say A csa/F, B simple F-alg, $f, g : B \to A$ alg. homs. Then f, g are conjugate (i.e. $\exists\, s \in A^\times$ s.t. $f(b) = s^{-1} g(b) s$ for all $b \in B$).

Special case: $B = A$

If A is a csa/F, every endo of A is an inner aut: $\exists\, s \in A^\times : f(a) = s^{-1} a s$.

(viz.: take $g = 1$ in S-N Thm)

Re pf of thm: By $\cdot \otimes A^{op}$, we're reduced to A a mx alg, since $A \otimes A^{op} = M_n(F) = End(F^n)$. Assume this. View F^n as B-module in two ways, via $f, g : B \to A$; say M_f, M_g. B simple $\Rightarrow$ M_f, M_g each iso to $\oplus$ of copies of unique minl B-module $\Rightarrow \exists\, s : M_f \xrightarrow{\sim} M_g$ over B; have $s \in GL_n(F) = A^\times$ is as claimed. ▬

Recall connections between quad. forms & quaternion algs:

$\left(\frac{a,b}{F}\right)$ is split $\Leftrightarrow$ $\langle a, b\rangle$ reps. 1;

norm on $\left(\frac{a,b}{F}\right)$ is q. form $\langle 1, -a, -b, ab\rangle$.

Can generalize to connection between quadratic forms & csa's, $\left(\begin{smallmatrix}C_{\ell}(V), \\ C\ell_0 V\end{smallmatrix}\right)$ by the Clifford algebra construction.

To motivate: For quat. alg's, view $\langle a, b\rangle$ as q.f. on 2-dim F-vs V with basis $\{i, j\}$. $q(\alpha i + \beta j) = a\alpha^2 + b\beta^2$ $(\alpha, \beta \in F)$

Can also view $\alpha i + \beta j \in \underset{\underset{V}{\cup\!\!|}}{A} = \left(\frac{a,b}{F}\right)$.

In A, compute

$$(\alpha i + \beta j)^2 = \alpha^2 a + \beta^2 b = q(\alpha i + \beta j)$$

i.e. $v^2 = q(v)$ for $v \in V$.

More generally, given a q.f. q on an F-v.s. V of dim n, can form an n^2-dim'l F-alg A containing V, st

$$\forall v \in \underset{A}{\overset{\cap\!|}{V}},\quad \underset{A}{\overset{\text{@}}{v^2}} \underset{(*)}{=} q(v) \in F \subseteq A$$

Build A by a universal construction:

Take $T(V) = \bigoplus_{n=0}^{\infty} T^n(V)$

tensor algebra of V over A

Let $I(q)$ = ideal gen by all elts of form $v \otimes v - q(v) \in T(V)$.

<u>Clifford algebra</u> $C(V,q) = T(V)/I(q)$

Satisfies $(*)$. For short: $C(V), C(q)$.

Ex $V = F^2$, $q = \langle a, b \rangle$ on V.

$\Rightarrow C(V,q) = \left(\frac{a,b}{F}\right)$. (as expected)

E.g. $q = h \rightsquigarrow C(V,h) = \left(\frac{-1,1}{F}\right) \cong M_2(F)$

Ex. V is 1-dim'l, $q = \langle a \rangle$.
V basis $\{t\}$. $q(t) = a$

Then $T(V) \cong F[t]$;

$I(q) = (t^2 - a)$,

$C(V,q) \cong F[t]/(t^2 - a)$

E.g. $F = \mathbb{R}$, $a = -1$: $C(V,q) \cong \mathbb{C}$

E.g. F arb, $a = 0$: $C(V,q) \cong F[t]/(t^2)$

Ex. $q = 0$ on V. $I(q)$ gen by all $v \otimes v$.

So $v \cdot v = 0$ in $C(V,q)$ for all $v \in V$

Get $C(V,q) \cong \Lambda^* V$

$[v_1 \otimes \cdots \otimes v_n] \longmapsto v_1 \wedge \cdots \wedge v_n$.

In gen'l, for $x, y \in V$, in $C(V,q)$

have $xy + yx = (x+y)^2 - x^2 - y^2$

$= q(x+y) - q(x) - q(y) = 2B(x,y)$

$\therefore xy = -yx \iff x \perp y$ wrt q.

Now take an orthog basis $x_1, \ldots, x_n$ for V wrt q

So $x_i \perp x_j$ for $i \neq j$, and $x_i x_j = -x_j x_i$.

So can write monomials in increasing order.

Since $x_i^2 = q(x_i) \in F$, all exponents ≤ 1.

So: $C(F, q)$ is spanned by the elts

$x_1^{e_1} \cdots x_n^{e_n}$, where each $e \in \{0, 1\}$.

$\therefore \dim C(V, q) \leq 2^n$.

$\uparrow$ will see: $=$.

Since $T(V) = \bigoplus_{n \geq 0}^{\infty} T^n(V)$ has a

$\mathbb{Z}$-grading, $C(V, q) = T(V) / I(q)$

has a $\mathbb{Z}/2$-grading.

mod $v^2 - q(v)$

in $T^2(V)$ in $T^0(V) = F$

$C(V, q) = C_0(V, q) \oplus C_1(V, q)$

image of $\bigoplus T^{2n}(V)$

image of $\bigoplus T^{2n+1}(V)$

(Like $\hat{W}(F) \rightsquigarrow W(F)$.)

Write ∂c for the $\mathbb{Z}/2$-degree of a homog. elt c.

($\mathbb{Z}/2$ grading)

Say $C(V, q)$ is a "super-algebra"

Ex. $V = F^2$, basis $\{i, j\}$, $q = \langle a, b\rangle$.

$C(V,q) = \left(\frac{a,b}{F}\right)$. $\dim = 4 = 2^2$. Gradings:

$C_0 = F \oplus kF$, $C_1 = iF \oplus jF$.

As a graded alg, write $\left\langle\frac{a,b}{F}\right\rangle$.

E.g. $q = \langle -1, 1\rangle$. $\left(\frac{-1,1}{F}\right) \xrightarrow{\sim} M_2(F)$

$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $i \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $k \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$C_0 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, $C_1 = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ $\quad \widehat{\mathrm{End}}(F^n)$ ''

For $M_2(F)$ with this grading, write $\hat{M}_2(F)$.

Ex. $V = F$, basis 1, $q = \langle a\rangle$. $\quad$ As gr. alg., write $F\langle\sqrt{a}\rangle$

$C(V,q) = F[t]/(t^2 - a)$; $C_0 = F\cdot 1$, $C_1 = F\cdot t$

So $\dim C(V,q) = 2$ here.