Recall: Given $a, b \in F^\times$, we can form the quaternion algebra $A = \left(\frac{a,b}{F}\right)$, a csa.

$\dim_F A = 4$; take subspace $V = \text{span}\{i, j\} \subset A$. On V, with basis $\{i, j\}$ we have the q.f. $q = \langle a, b \rangle$.

$v \in V \subset A$, $v = (\alpha, \beta) = \alpha i + \beta j$ satisfies

$$q(v) = a\alpha^2 + b\beta^2 \in F \subset A$$

$$\quad\quad\quad\quad \overset{\shortparallel}{v^2} \in A$$

More generally, given a q.f. q on a v.s. V over F, we can form an algebra $A = C(V, q)$ over F st $\forall v \in V \subseteq A$,

$$\underset{F}{\underset{\cap}{q(v)}} = \underset{A}{\underset{\cap}{v^2}}, \quad F \subseteq A$$

— the Clifford algebra.

Is $A = C(V, q)$ always a csa?

Answer: No. But it's always a csga, $A = A_0 \oplus A_1$

$\mathbb{Z}/2$-graded deg 0 deg 1 parts

If $\dim q$ is even, then A is a csa.

If $\dim q$ is odd, then A_0 is a csa.

Just as equiv. classes of csa's $/F$ form the Brauer group, $Br(F)$, equiv classes of csga's $/F$ form the Brauer-Wall group, $BW(F)$.

Here, $Br(F) \leq BW(F)$, and in fact

$$0 \to Br(F) \to BW(F) \to Q(F) \to 0$$

is exact, where $Q(F)$ is as before.

More holds: Comm. diag. with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & I^2(F) \arrow[r] \arrow[d, "\gamma"] & W(F) \arrow[r] \arrow[d, "\Gamma"] & W(F)/I^2(F) \arrow[r] \arrow[d, "s"] & 0 \\
0 \arrow[r] & Br(F) \arrow[r] & BW(F) \arrow[r] & Q(F) \arrow[r] & 0
\end{tikzcd}$$

Here Γ and $\gamma = \Gamma / I^2(F)$ are given by taking the Clifford algebra.

Also: $\ker \gamma = \ker \Gamma = I^3(F)$,

and $\quad \mathrm{Im}\, \gamma = Br(F)[2]$ (subgp of $Br(F)$ gen by quat. algs)

So $I^2(F) / I^3(F) \cong Br(F)[2]$.

To understand the structure of $BW(F)$ in terms of $Br(F)$, use the s.e.s.

$$0 \to Br(F) \to BW(F) \to Q(F) \to 0$$

Re structure of $Q(F)$, recall

$$1 \to F^\times / F^{\times^2} \to Q(F) \to \mathbb{Z}/2 \to 0$$

and an elt $\xi \in Q(F)$ is of the form (e, d) with $e \in \mathbb{Z}/2$, $d \in F^\times / F^{\times^2}$.

Can similarly describe an elt of $BW(F)$ by a pair (D, ξ), where $D \in Br(F)$ and $\xi \in Q(F)$.

(See Lam, pp. 115-116 for details.)

Writing $\xi = (e, d) \in Q(F)$, we can then write an elt of $BW(F)$ as a triple $(D, \underbrace{e, d}_{\xi})$.

We can then explicitly work out the mult law on $BW(F)$ in these terms:

Thm (Lam, Chap V, Th 3.8)
Given (D, ξ), $(D', \xi') \in BW(F)$, where $D, D' \in Br(F)$, $\xi = (e, d)$, $\xi' = (e', d') \in Q(F)$,

$$(D, \xi) \cdot (D', \xi') = \left(D \cdot D' \cdot \left(\frac{d, (-1)^{e+e'} d'}{F}\right), \xi \xi'\right)$$

and $(D, \xi)^{-1} = (D^{-1} \cdot A, \xi^{-1})$ where

$$A = \begin{cases} \left(\frac{d, d}{F}\right) & \text{if } e = 0 \\ 1 & \text{if } e = 1. \end{cases}$$

Here we write $Br(F)$ multiplicatively.
(Recall: The group law on $Q(F)$ is given by: $(e, d) \cdot (e', d') = (e+e', (-1)^{ee'} dd')$.)

A motivation for Clifford algebras: to associate to each q.f. a csa.
But: we **don't** always get a csa, just a csga A, depending on $\dim q$.
To remedy this: recall if $\dim q$ is even then $A = C(q)$ is a csa; otherwise, $A_0 = C_0(q)$ is a csa, where $A = A_0 \oplus A_1$ (graded pieces)

So: to each q.f. q, associate a csa: $c(q) = \begin{cases} C(q) & \text{if } \dim q \text{ is even} \\ C_0(q) & \text{if } \dim q \text{ is odd} \end{cases}$

Can view $c(q) \in Br(R)$.

Because of the two cases, $c: W(F) \to Br(F)$ is not a hom; but it is on $I^2(F)$.

forms are even, so $c = \gamma = \Gamma$, i.e. $c = C$ there.

Terminology for the class of $C(q)$, $c(q)$:

In Lam: $C(q)$: <u>Clifford invariant</u> of q

$c(q)$: <u>Witt invariant</u> " "

Some others: $C(q)$: <u>Clifford algebra</u> of q

$c(q)$: <u>Clifford invariant</u> " "

A related invariant of a q.f. q:

$s(q)$: the <u>Hasse invariant</u> of q.

(Also called: Hasse-Witt invariant, Hasse symbol, 2° Stiefel-Whitney class)

If $q = \langle a_1, \dots, a_n \rangle$, define $s(q) = \prod_{i<j} \left(\frac{a_i, a_j}{F} \right) \in Br(F)$.

Using chain equivalence and computations involving iso of quaternion algs, one can show

Thm (Lang Chap V, Prop. 3.18)
If q, q' are isometric diagonal qf's, then $s(q) = s(q') \in Br(F)$.

So we get a well defined map
$$s : \hat{W}(F) \longrightarrow Br(F).$$

This is not a group hom. But can modify to get a group hom:
$$\hat{W}(F) \longrightarrow BW(F)$$
$$q \longmapsto (\underset{\in Br(F)}{s(q)}, \overset{\mathbb{Z}/2}{\underset{\cup}{0}}, \underset{\in F^\times/F^{\times 2}}{\det(q)})$$

(The verification uses the explicit form above of mult. on $BW(F)$.)

Ex. $q = \langle a, b \rangle$. Form of dim 2. Here
$$s(q) = \left(\frac{a, b}{F}\right).$$

In this example, $C(q) = \left\langle \frac{a,b}{F} \right\rangle$ as a graded alg, and this is a csa because $\dim q = 2$ is even.

viz $\left(\frac{a,b}{F}\right)$. Also $c(q) = C(q)$.

So here, $c(q) = s(q)$.

Ex. $q = \langle a \rangle$, $\dim = 1$. Then $C(q) = F = s(q)$.

More generally:

Thm (Lam, Chap. V, Prop 3.20)

Say $\dim q = n$.

If $n \equiv 1$ or $2 \bmod 8$, then $C(q) = S(q)$.

Otherwise, $c(q) = s(q)\left(\frac{-1,a}{F}\right) \in Br(F)$

where $a \in F^\times$ depends on $n \bmod 8$:

$$a = \begin{cases} -\det(q) & \text{if } n \equiv 3,4 \pmod 8 \\ -1 & \text{if } n \equiv 5,6 \pmod 8 \\ \det(q) & \text{if } n \equiv 7,8 \pmod 8 \end{cases}$$

The proof is explicit, and uses induction on n. (8 cases)

Recall: Two binary quadratic forms $q = \langle a, b \rangle$ and $q' = \langle a', b' \rangle$ are isometric $\iff \det q = \det q'$ and $\left(\frac{a,b}{F}\right) \cong \left(\frac{a',b'}{F}\right)$.

Question: Does this generalize?

Above, $\left(\frac{a,b}{F}\right) = c(q) = s(q) \in Br(F)$

What if q is not necessarily binary?

Thm (Lam, Chap V, Thm 3.21)

If $\dim q = \dim q' \leq 3$ then TFAE:

1) $q \cong q'$
2) $\det q = \det q'$ and $c(q) = c(q')$
3) $\det q = \det q'$ and $s(q) = s(q')$.

Re pf: $(1) \Rightarrow (2), (3)$ trivially.

$(2) \iff (3)$ by above result relating $c(q)$, $s(q)$

For $(3) \Rightarrow (1)$, proof is explicit, using quaternion alg's + the fact that quat. alg's are iso $\iff$ have isometric norm forms.

Using the above results then get a classification of q.f. when u-invariant is small:

Thm (Lam, Chap V, Prop 3.25)

Suppose $u(F) \leq 4$.

i.e., every qf/F of dim ≥ 5 is isotropic

Let q, q' be q.f.'s over F.

Then: $q \cong q' \iff \dim q = \dim q'$,

$\det q = \det q'$,

and $s(q) = s(q')$.

Proof. ($\Rightarrow$) is clear.

($\Leftarrow$): Let $n = \dim q = \dim q'$

Case 1: $\dim = n \leq 3$. Done by above result.

Case 2: $\dim = n \geq 4$. So $\dim q = \dim q' \geq u(F)$.

$\therefore q, q'$ are universal; so they represent 1. Write $q \cong \langle 1 \rangle \perp \varphi$, $q' \cong \langle 1 \rangle \perp \varphi'$. So $\dim \varphi = \dim \varphi'$ and $\det \varphi = \det \varphi'$, by these facts for q, q'.

Since $\dim \varphi = \dim \varphi' < n$, by the inductive hypothesis, STS $s(\varphi) = s(\varphi')$.

WMA $q = \langle \underset{\overset{\|}{1}}{a_1}, \underbrace{a_2, \ldots, a_n}_{\varphi} \rangle$. Then

$$s(q) = \prod_{i<j} \left(\frac{a_i, a_j}{F}\right) = \prod_{j=2}^{n} \left(\frac{1, a_j}{F}\right) s(\varphi)$$

$$= \left(\frac{1, a_2 \cdots a_n}{F}\right) s(\varphi)$$

$$= \left(\frac{1, \det \varphi}{F}\right) s(\varphi) \in Br(F)$$

Similarly for $s(q')$. But $s(q) = s(q')$ and $\det \varphi = \det \varphi'$. $\therefore$ $s(\varphi) = s(\varphi')$.
So $\varphi \cong \varphi'$ by the inductive hypothesis; & so $q \cong \langle 1 \rangle \perp \varphi \cong \langle 1 \rangle \perp \varphi' \cong q'$. —

Some related topics:

Periodicity of Clifford alg's $(C\ell V, \delta q)$:

If $F = \mathbb{R}$ and q is a regular q.f., then $q \cong \langle \underbrace{-1, \ldots, -1}_{a}, \underbrace{1, \ldots, 1}_{b} \rangle$

For <u>any</u> F (of char $\neq 2$), if q is of this form write $C^{a,b} = C(q)$, Clifford alg. of q.

<u>Prop</u> (Lam, Ch V, Prop 6.1):

$$C^{a+n,b+n} \cong \hat{M}_{2^n}(C^{a,b})$$

<u>Proof</u> $C^{a+n,b+n} = C^{a,b} \hat{\otimes} C^{n,n}$

$$= C^{a,b} \hat{\otimes} C(nh)$$

$$= C^{a,b} \hat{\otimes} \hat{M}_{2^n}(F)$$

$$= \hat{M}_{2^n}(C^{a,b}).$$

So to study $C^{a,b}$, we're reduced to studying $C^{a,0}$, $C^{0,b}$ for $a,b \geq 0$.

Also, $C^{a,b}$ depends only on a,b mod 8.

(See Lam, Chap V, Prop 4.2)

Using this, can express all $C^{a,b}$ in terms of $C^{1,0}$, $C^{2,0}$, $C^{0,1}$, $C^{0,2}$, and $\otimes$ and $\hat{M}_{2^i}$.

Here $C^{1,0} = C(\langle -1 \rangle) = F\langle \sqrt{-1} \rangle$

$$C^{2,0} = C(\langle -1,-1 \rangle) = \left\langle \frac{-1,-1}{F} \right\rangle$$

$$C^{0,1} = C(\langle 1 \rangle) = F\langle \sqrt{1} \rangle$$

$$C^{0,2} = C(\langle 1,1 \rangle) = \left\langle \frac{1,1}{F} \right\rangle.$$

E.g. $C^{3,0} \cong C^{2,0} \hat{\otimes} C^{1,0} \cong C^{2,0} \otimes C^{0,1}$

See the chart on p. 123 of Lam for all cases.

Composition of quadratic forms (Lam, Ch V, §5)

Recall: Using norm forms on $F[\sqrt{-1}]$ and on $\left(\frac{-1,-1}{F}\right)$, we saw:

- a product of two elements of the form $x_1^2+x_2^2$ is also of this form.
- a product of two elements of the form $x_1^2+x_2^2+x_3^2+x_4^2$ is also of this form.

More generally?
And given $m,n>0$, is there a formula

$$(x_1^2+\cdots+x_m^2)(y_1^2+\cdots+y_n^2) = z_1^2+\cdots+z_n^2, \quad (*)$$

where each z_i is a homogeneous bilinear form in $(x_1,\dots,x_m;y_1,\dots,y_n)$? Using Clifford alg's:

Thm (Lam, Chap V, Th 5.11) [Radon]
Let $F=\mathbb{R}$, write $n=2^c n_0$ with n_0 odd, and write $c=4a+b$ with $0\le b\le 3$.
Then there is a formula $(*)$ iff $m\le 8a+2^b$. (E.g. OK if $m=n=8$, not if $m=n=16$)

Back to $I^2(F)/I^3(F)$.

$$\begin{array}{c} I^2(F)/I^3(F) \\ \wr\big\downarrow \gamma \\ Br(F)[2] \end{array}$$

given by Clifford alg.

Recall $Br(F)[2]$ is generated by quaternion algebras $\left(\frac{a,b}{F}\right)$.

These satisfy:

1) Bimultiplicativity:

$$\left(\frac{a,b}{F}\right)\otimes\left(\frac{a',b}{F}\right) = \left(\frac{aa',b}{F}\right) \in Br(F)[2]$$

and similarly with roles of a,b reversed

2) $\left(\frac{a,1-a}{F}\right)$ is trivial in $Br(F)[2]$

3) Symmetry in a,b

4) Elements are 2-torsion.

Since $\gamma: I^2/I^3 \longrightarrow Br(F)[2]$ is iso, the same hold for I^2/I^3.

(Can also verify directly – see Lam, Chap V, Prop. 6.5 (2).)

This suggests forming an abstract object with these properties.

For this, take $F^\times \otimes_{\mathbb{Z}} F^\times$; gen by $a \otimes b$; this satisfies (1) above (which is bilinearity if write additively).

Mod out by the subgroup gen by all elts $a \otimes (1-a)$. Get a group $K_2(F)$.

Write $[a,b]$ for the class of $a \otimes b$.

To make 2-torsion, take the quotient

$$k_2(F) := K_2(F)/2$$

ie mod squares (if write mult'ly)

This turns out to be symmetric because in $K_2(F)$, it's anti-symmetric: $[a,b]^{-1} = [b,a]$ (Lam, Chap V, Lemma 6.3(2))

So get a commutative diagram

$$\begin{tikzcd}
\langle 1,-a,-b,ab\rangle & {[a,b]} \arrow[l, mapsto, bend right] \arrow[r, mapsto, bend left] & \left(\frac{a,b}{F}\right) \\
& k_2(F) \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
I^2(F)/I^3(F) \arrow[rr, "\gamma", "\sim"'] & & Br(F)[2]
\end{tikzcd}$$

Here γ, as before, is given by taking the Clifford invariant, and is an isomorphism.

α takes $[a,b]$ to $\langle 1,-a\rangle \otimes \langle 1,-b\rangle$

$$\begin{aligned} &\langle 1,-a\rangle \otimes \langle 1,-b\rangle \\ &\quad \| \\ &\langle 1,-a,-b,ab\rangle \\ &\quad \| \\ &\text{norm form of } \left(\frac{a,b}{F}\right). \end{aligned}$$

β takes $[a,b]$ to $\left(\frac{a,b}{F}\right)$.

Here α is an iso

(see Lam, Chap V, Thm 6.7). The proof uses Chain equivalence and the Hasse invariant $s(q)$.

Since the diagram is commutative, β is also an iso.

So $I^2(F)/I^3(F) \cong Br(F)[2] \cong k_2(F)$

This describes both I^2/I^3 and $Br(F)[2]$, since $K_2(F)$ can be studied:

Ex. If F is finite, $K_2(F)$ is trivial.
([Lam, Chap V, Ex. 6.14;
due to Steinberg.)

Ex. $K_2(\mathbb{Q}) = \bigoplus_{p \text{ prime}} A_p$
where $A_2 = \{\pm 1\}$, $A_p = (\mathbb{Z}/p)^\times$.
odd prime
(Due to Tate.)
This turns out to be equivalent
to Quadratic Reciprocity
in number theory!
So this is sometimes called
"The Gauss-Tate Theorem."

Re K_2: this is part of a collection
of groups K_n, due to Milnor.

Using those, we can generalize
the above.