<u>Recall</u>: For G an algebraic group over a field F,

$$H^1(F,G) \xleftrightarrow{\text{bij}} \{\text{iso classes of } G\text{-torsors}/F\}$$

and for E/F a Galois field extension,

$$H^1(E/F,G) \xleftrightarrow{\text{bij}} \left\{\begin{array}{l}\text{iso classes of } G\text{-torsors}/F \\ \text{that become trivial } /E\end{array}\right\}$$

If Δ is a (functorial) algebraic structure over F, and $G = \mathrm{Aut}(\Delta)$, then

$$H^1(E/F,G) \xleftrightarrow{\text{bij}} \left\{\begin{array}{l}\text{iso classes of objects}/F \text{ that} \\ \text{become iso to } \Delta \text{ over } E\end{array}\right\}$$

and if all objects become iso $/F^{sep}$ then

$$H^1(F,G) \xleftrightarrow{\text{bij}} \{\text{iso classes of objects } /F\}$$

Ex. 1) q.f./F, $\Delta = q$ $\rightsquigarrow$ $G = O(q)$

2) q.f. /F of det δ, $\Delta = q$ $\rightsquigarrow$ $G = SO(q)$

3) csa /F of dim n, $\Delta = M_n(F)$ $\rightsquigarrow$ $G = PGL_n$

For F a global field, an F-variety V satisfies a local-global principle if
$$V(F_v) \neq \emptyset \text{ for all } v \Rightarrow V(F) \neq \emptyset.$$

If V is a G-torsor over F, then this says: V trivial / all $F_v \Rightarrow V$ trivial / F.

The obstruction to a LGP holding for all G-torsors / F is
$$Ш(F,G) := \ker\left(H^1(F,G) \to \prod_v H^1(F_v,G)\right).$$

I.e.: LGP holds for all G-torsors / F
$$\Updownarrow$$
$Ш(F,G)$ is trivial

So given an algebraic structure Δ over F with $G = \mathrm{Aut}(\Delta)$, we have that a LGP holds for $\Delta \Leftrightarrow Ш(F,G) = 1.$

Ex. Hasse–Minkowski $\Leftrightarrow Ш(F, O(q)) = 1$

Ex. F a # fld, G a rational conn. lin. alg. gp.
$\Rightarrow$ Ш$(F, G) = 1$; and $SO(q)$ is rat'l & conn;
so given a regular qf q of $\det = \delta$, we
get a LGP for a qf of $\det = \delta$ to be iso to q.

Relationship between LGP's & numerical field invariants:

Ex. u-invariant. We saw: u-invariant of a non-archimedean local field is 4.

So: Hasse-Minkowski $\Rightarrow$
u-invariant of a global fn fld,
or of a totally imag. # fld, is also 4.

(For a # field F with a real embedding, we have $u(F) = \infty$. But the Elman-Lam version satisfies $u'(F) = 4$.)

Ex. period-index problem.

Recall: For a csa A over F, given an elt $\alpha \in Br(F)$,

$per(\alpha)$ = order of α in $Br(F)$

$ind(\alpha)$ = degree of D, where $A = M_n(D)$.

$per(\alpha) \mid ind(\alpha)$, + same primes divide each.

For a local field F, if F is non-archimedean,

$Br(F) = \mathbb{Q}/\mathbb{Z}$ and $Br(F)[n] = \frac{1}{n}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$

(e.g. see Pierce, Associative Algebras, Thm 17.10),

while $Br(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$. For both cases:

$$per = ind.$$

By thm of Albert-Brauer-Hasse-Noether (LGP for csa's), get per = ind for global fields.

The above results focus on global fields:
flds or fn flds of curves / finite fields

Higher dim'l analogs?

Ex. $\mathbb{Q}(x)$, or $\mathbb{F}_p(x,y)$?
— fn fld of a curve over a global field.

Local-global principles?

u-invariant?

Relationship of period to index?

e.g. $\mathbb{Q}(x)$

To study the fn fld of a curve over a global field,
could first study - - - - - - - - - local field.
— e.g. $\mathbb{Q}_p(x)$, or $\mathbb{F}_p((t))(x)$

There can ask the same questions. More generally,
let K be any cdvf, & F = fn fld of a K-curve.
F has a local & a global aspect:

E.g. $\mathbb{F}_p \underbrace{((t))}_{\text{local part}} \underbrace{(x)}_{\text{global part}}$

— call F a **semi-global field**.

What to expect for u-invariant?

Field k	$u(k)$	$u(k(x))$
alg. closed	1	2
finite	2	4
non-arch local (e.g. $\mathbb{Q}_p$, $\mathbb{F}_p((t))$)	4	?
$\mathbb{Q}_p((t))$	8	?
$\vdots$	$\vdots$	$\vdots$

cdvf

by Springer's Thm

Semi-global

Pattern suggests: $u(k)$ is always a power of 2. Was a conjecture of Kaplansky (who defined u-invariant).

Merkurjev found example with $u(k) = 6$ (1988; Lam, Chap XIII, §2)

But $u(k)$ is never 3, 5, 7 (Lam Ch XI, Prop 6.8).

Other odd? Izhboldin found an example with $u = 9$ (Annals of Math, 2001).

Q: For "reasonable" fields, is $u(k)$ always a power of 2?

Above chart suggests that $u(\mathbb{Q}_p(x)) = 8$.
Wasn't even known to be *finite*,
until 1998. Several results then (for $p \neq 2$):

- Merkurjev: $u(\mathbb{Q}_p(x)) \leq 26$.
- van Geel - Hoffman: $u(\mathbb{Q}_p(x)) \leq 22$.
- Parimala - Suresh: $u(\mathbb{Q}_p(x)) \leq 10$.

— Later, Parimala - Suresh showed: $u(\mathbb{Q}_p(x)) = 8$.
(arXiv 2007, Annals of Math 2010)

Two other proofs of this:

— Harbater - Hartmann - Krashen, Inventiones 2009
(arXiv 2008)

- also showed other u-invariant results, including $u(\mathbb{Q}_p((t))(x)) = 16$ (as suggested by chart)

— Leep, Crelle 2013

- also showed $u(\mathbb{Q}_p(x_1, \ldots, x_n)) = 2^{n+2}$,
and allowed all p (including 2)

For fields like $\mathbb{Q}_p(x)$:

What to expect for period-index relationship?

Recall: for a finite field F,

$Br(F)$ is trivial, so per, ind are trivial.

For F a non-arch. local field, e.g. $\mathbb{Q}_p$

or a global field, per = ind. e.g. $\mathbb{Q}$, $\mathbb{F}_p(x)$

In general, $\operatorname{per}\alpha \,|\, \operatorname{ind}\alpha$ for $\alpha \in Br(F)$,

and $\forall \alpha\ \exists n$: $\operatorname{ind}\alpha \,|\, (\operatorname{per}\alpha)^n$.

For a "reasonable" F, is there a uniform n?

— e.g. for $F = \mathbb{Q}_p(x)$, $\mathbb{Q}_p((t))(x)$?

Ans: For α st $p \nmid \operatorname{per}\alpha$,

in $Br(\mathbb{Q}_p(x))$ have $\operatorname{ind}\alpha \,|\, (\operatorname{per}\alpha)^2$

in $Br(\mathbb{Q}_p((t))(x))$ have $\operatorname{ind}\alpha \,|\, (\operatorname{per}\alpha)^3$

etc.

Two different proofs:

- Lieblich, Crelle 2011 (arXiv, 2007)
- HHK, Inventiones 2009 (arXiv 2008) as above

In the HHK paper — the proofs for the u-invariant & for period-index were in parallel — & both relied on LGP's

— analogously to how the results for global fields on u + per-ind follow from LGP's.

Here — consider semi-global fields, i.e. finite extensions F of $K(x)$ where K is a c.d.v.f.

$\quad$ Ex. $F = \mathbb{Q}_p(x)$ or $k((t))(x)$. a field

As for global fields, we can consider the set of absolute values v on F ($\leftrightarrow$ discr. val's) and consider LGP's wrt those.

Ex. If V is a variety over a g.f. F, say V satisfies a LCP $/F$ if:

$$V(F_v) \neq \emptyset \text{ for all } v \Rightarrow V(F) \neq \emptyset.$$

In particular, if V is a G-torsor $/F$ for some algebraic group G over F, a LCP for F says:

$$V \text{ is trivial}/F \iff V \text{ is trivial}/\text{each } F_v \quad (*)$$

Given G over F, can ask:

Do all G-torsors $/F$ satisfy a LCP?

As before, the obstruction to $(*)$ is

$$Ш(F,G) := \ker\left(H^1(F,G) \to \prod_v H^1(F_v,G)\right)$$

As before, can ask:
if G is a rational connected lin. alg. gp. $/F$, g.f. is there a LCP?

Answer: Yes. (HHK)

So get LGP's for various alg objects $/F$.

Ex. Take $\delta \in F^\times$. If q, q' are reg. q.f.'s/F of dim $= n$ and $\det = \delta \in F^\times/F^{\times 2}$, and if $q \cong q'$ locally, then $q \cong q'$ over F.

— pf as for # flds, using $H^1(F, SO(q))$ classifies q.f./F of $\det = \delta \in F^\times/F^{\times 2}$.

conn. lin. alg. group

As a consequence, as for global fields, we get: q hyperbolic locally $\Rightarrow$ q hyperbolic/F

Even more: have LCP for homogeneous spaces X that aren't necessarily torsors.

principal homog. sp.

— via G acts on X over F, st $\forall E/F$, $G(E)$ acts transitively on $X(E)$
(ext. flds) (but not nec. simply trans.)

Ex. Let q be a regular q.f. of $\dim = n > 2$.

Let $Q \subset \mathbb{P}^{n-1}_F$ be the projective hypersurface defined by $q = 0$. Then $O(q)$ acts transitively on Q, by the Witt extension thm.

b/c $\dim q > 2$, $\dim Q > 0$

Here Q is connected, but $O(q)$ consists of two conn. components: $SO(q)$ + its coset. So $SO(q)$ acts trans. on Q.

b/c orbit of a pt is open + closed

but not simply trans; not a torsor.

$SO(q)$ conn + rat'l. So the above LGP for homog. spaces applies, + get:

Q has an F-pt iff it has points locally;

i.e. q isotropic / F iff it is isotropic locally.

So have a Hasse-Minkowski Thm for qf's of dim > 2 over a s.g.f. F.

What about binary quadratic forms?

Still true if $F = K(x)$ (e.g. $\mathbb{Q}_p(x)$).

But false for some sgf's F,

e.g. $F = \mathbb{Q}_p(x)[\sqrt{\alpha}]$, where

$$\alpha = x(x-1)(1-px),$$

Issue here: Chebotarev Density fails for this field F:

every prime splits in $F[\sqrt{x(x-1)}]$.

From the above Hasse-Minkowski Thm, can get
$u(\mathbb{Q}_p(x)) = 8$, $\quad u(\mathbb{Q}_p((t))(x)) = 16$, ...

How to get these LGP's over sgf's?

First:

For a semi-global field F, (fn fld of curve / cdvf K, i.e. fin extn of $K(x)$.)
there are several possible LGP's to consider.

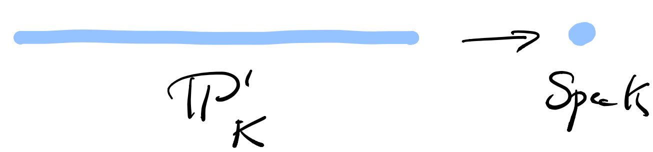
As above, can consider LGP wrt discrete val's on F.

Another possibility:
For simplicity, first take $F = K(x)$,
K a cdvf, $\mathcal{O}_K$ = assoc cdvr, $\supset \mathfrak{m}$ = maximal ideal, $k = \mathcal{O}_K/\mathfrak{m}$ res. fld.

Ex $K = \mathbb{Q}_p$, $\mathcal{O}_K = \mathbb{Z}_p$, $\mathfrak{m} = (p)$, $\underset{\text{res fld}}{k} = \mathbb{F}_p$

Ex $K = k((t))$, $\mathcal{O}_K = k[[t]]$, $\mathfrak{m} = (t)$, res fld k.

Can view F as fn fld of $\mathbb{P}^1_K$, proj x-line $/K$



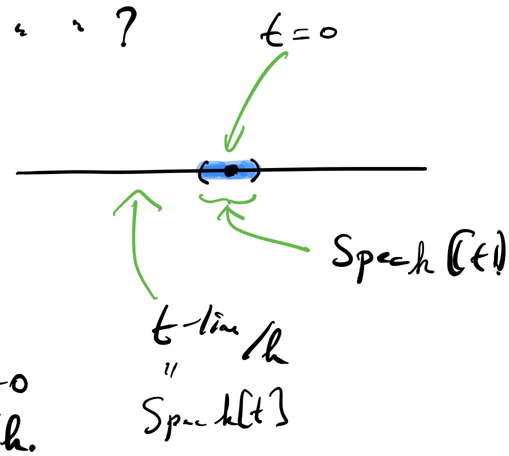
Can also view K as frac $\mathcal{O}_K$, and F as fn fld of $\mathbb{P}^1_{\mathcal{O}_K}$: proj line $/\mathcal{O}_K$

equiv., $/\mathrm{Spec}\,\mathcal{O}_K$.

What does $\mathbb{P}^1_{\mathcal{O}_K}$ look like?

" " $\mathrm{Spec}\,\mathcal{O}_K$ " " ?

Ex. $\mathcal{O}_K = k[[t]]$.

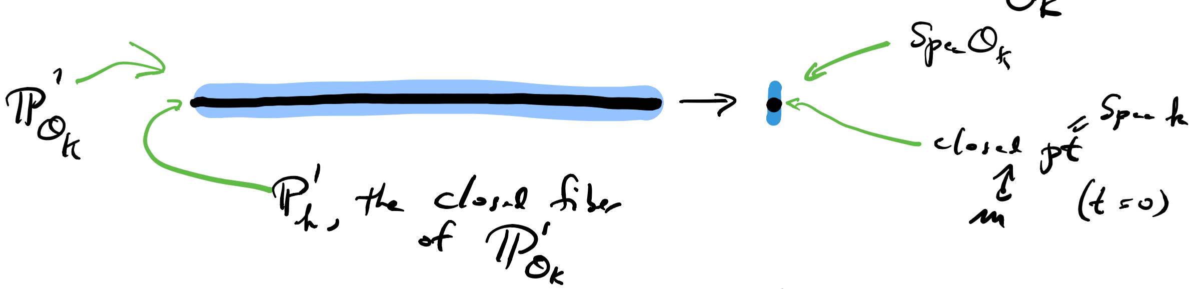


= completion of the local ring at the point $t=0$ on the t-line $/k$.

$\mathcal{O}_K$ has 2 prime ideals: $\mathfrak{m} = (t)$ and $0 = (0)$

Spec $k \longleftrightarrow$ closed pt of $\mathrm{Spec}\,\mathcal{O}_K$: $t=0$; generic pt of $\mathrm{Spec}\,\mathcal{O}_K$: $t \neq 0$.

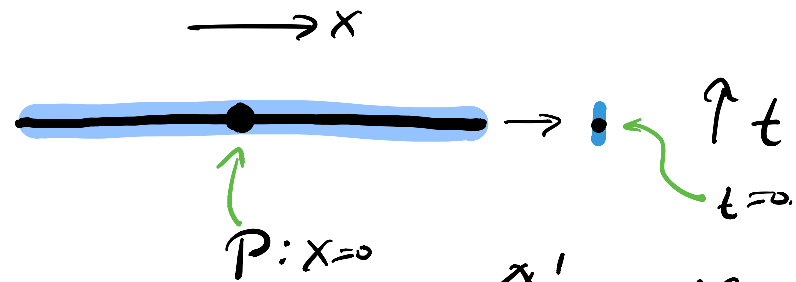
From this point of view, F is fn fld of $\mathbb{P}^1_{\mathcal{O}_K}$:



On the closed fiber $\mathbb{P}^1_k \overset{\text{open}}{\supset} \mathbb{A}^1_k \ni$ a pt P : corresponds to a prime ideal of $k[x]$

$(t=0)$

Ex. Take the pt $P: x=0$ on the closed fiber



$$\mathbb{A}^1_k \leftrightarrow k[x]$$
$$\cap$$

On the closed fiber $\mathbb{P}^1_k$, $P \leftrightarrow$ max ideal $(x) \subset k[x]$

$$\mathbb{A}^1_{\mathcal{O}_K} \leftrightarrow \mathcal{O}_K[x] = k[[t]][x].$$
$$\cap \qquad\qquad \Downarrow$$

In $\mathbb{P}^1_{\mathcal{O}_K}$, $\tilde{P} \leftrightarrow$ max. ideal $(x,t) \subset k[[t]][x]$

The completion of the local ring of $k[[t]][x]$ at (x,t)

(= - - - - - - - - - - - - - - - $\mathbb{P}^1_{\mathcal{O}_K}$ at P)

is $k[[x,t]] = \hat{\mathcal{O}}_{\mathbb{P}^1_{\mathcal{O}_K},P}$; a 2-dim'l complete local ring.

(so not a dvr) e.g. $(0) \subset (t) \subset (x,t)$

Its fraction field is written $k((x,t)) =: F_P$.

In general, for every pt P on closed fiber of $\mathbb{P}^1_{\mathcal{O}_K}$, we get an associated field F_P.

$$F_P$$
$$\cup$$
$$F$$

Summary: Take a semi-global field F, the function field of a projective curve C over a cdvf K; or equiv. of a proj. curve $\mathcal{X}$ over the cdvr $\mathcal{O}_K$. For each pt P on the closed fiber $X \subset \mathcal{X}$ (a curve $/ k = \mathcal{O}_K/\mathfrak{m}$), we get a field F_P via completion + frac.

Have $F \subset F_P$ for all P.

Can then ask for a LGP for F wrt the fields F_P, instead of the F_v's.

frac of 2-dim complete local rings

frac of 1-dim complete loc. rings

Relationships:

Every F_P is contained in many F_v's, and every F_v contains an F_P.

Can define:

$$Ш_{\mathcal{X}}(F, G) = \ker\Big(H^1(F, G) \to \prod_{P \in X} H^1(F_P, G)\Big)$$

So $Ш_X(F,G)$ is the obstruction to the LGP for G-torsors wrt F_P's:

If there is an F_P-pt for all P, is there an F-pt.

Have $Ш_X(F,G) \subseteq Ш(F,G)$.

Turns out: to get results about numerical invariants (u-invariant, period-index), it suffices to consider this LGP, which is easier to study.

HHK: For G a rat'l conn lin. alg. gp. $/F$, sgf, $Ш_X(F,G)$ is trivial.

$\rightsquigarrow$ LGP for q. forms + csa's wrt F_P's.

CPS: In context of q. forms + csa's, even the corresponding obstructions $Ш(F,G)$ are trivial.

Colliot-Thélène, Parimala, Suresh