Surface Classification and Geography

An Essay on the Classification, Geography and Moduli Spaces of Complex Surfaces of General Type

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Contents

1 Introduction

The intuitive classical problem in Algebraic Geometry is to classify all algebraic varieties up to isomorphism. This problem is very difficult even for surfaces, as it could be expected. Birational equivalence, a coarser equivalence relation between algebraic varieties leads to a coarser, simpler, but still difficult and currently incomplete classification as well as a step towards the fine classification.

We are interested in the classification of complex algebraic surfaces, that is algebraic varieties of dimension two over $\mathbb{C}$. For this purpose, the Enriques classification of surfaces, stated below, gives us a broad classification into several classes, each of which have been more or less well understood, apart from the class of surfaces of general type.

In section 1, we discuss the basic terminology and setting of the birational geometry of algebraic surfaces and state the Enriques classification of algebraic surfaces. This can be seen as a general introduction to the study of algebraic surfaces.

In section 2, we consider surfaces of general type. We begin with the fact that the Chern invariants occur in a clear-cut region of the plane and the geography problem and mention current progress towards it. We then consider moduli spaces for surfaces of general type, these do exist by a result of Gieseker for which we illustrate the main strategy. Finally, we consider some results on the geometry of the Gieseker schemes and deformations of surfaces of general type.

Even though we have done our best to assume as little as possible. It has not been possible to give a complete exposition of all the facts and concepts or pursue full generality, due to the number of different fields coming together in the study of algebraic surfaces, as well as restrictions of space and time. We have resorted to referring the reader to relevant sources when necessary.

1.1 Birational Geometry, the Picard Group and Minimal Surfaces

A rational map $f : S_1 \dashrightarrow S_2$ is a an equivalence class of maps, each from an open dense subset of S_1 into S_2 under the equivalence relation $f_1 \sim f_2$ iff f_1 and f_2 agree on the intersection of their domains.

Definition 1.1 *A birational map is a rational map with a rational inverse. Two varieties S_1 and S_2 are said to be birationally equivalent if there is a birational map between them.*

Apart from the general argument that two varieties having open and dense subsets that are the same would be very similar, there are three other reasons why birational equivalence is very convenient,the first is the following elementary fact,

Theorem 1.2 *Let V and W be two varieties over a field k. $k(V) \cong k(W)$ if and only if V is birational to W.*

By a birational morphism, we shall mean a birational map $f : S_1 \dashrightarrow S_2$, such that f is defined on all of S_1. As an example of two surfaces that are birationally equivalent but not isomorphic, one can consider $\mathbb{P}^2$ and the variety given by the homogeneous polynomial $x_1x_2 - x_3x_4 = 0$ in $\mathbb{P}^3$.

An important example of a birational equivalence is the blow-up of a surface at a point. Let S be a variety and $p \in S$, by "blowing up" at p, we obtain another surface S' and a map $\epsilon : S' \longrightarrow S$ such that the restriction of ϵ to $S' - \epsilon^{-1}(p)$ is an isomorphism onto $S - p$ and $\epsilon^{-1}(p) \cong \mathbb{P}^1$. For an explicit construction, see [20]. As mentioned above, birational maps are not complicated maps, they factor out through blowups and blowing down, this is the second reason why birationality is convenient. [3]

Proposition 1.3 *Let $\phi : S \dashrightarrow S'$ be a birational map between two surfaces, then there is a surface X and maps f and g which are compositions of a finite number of blowups and isomorphisms such that $\phi \circ f = g$*

$$\begin{tikzcd}
 & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\
S \arrow[rr, dashed, "\phi"'] & & S'
\end{tikzcd}$$

The third reason why birationality is convenient is that its generalisation into positive dimensional subvarieties, the monoidal transformation (but these are also called blowups in some sources) can be used to resolve singularities. So every surface is birationally equivalent to a smooth one. In general, singularities of higher-dimensional varieties can also be resolved, through σ-processes, which is Hironaka's theorem [14], [15]. From here on, by a surface will mean a smooth algebraic surface. Another consideration is the one of projectivity; that is if there exist surfaces (as two-dimensional abstract varieties) that cannot be embedded in projective space. Through considering the union of bisecants, and the union of tangents to a surface S, which are parameterized by the complement of the diagonal in $S \times S$ and the projective tangent bundle to S respectively, and through projecting away from a point to reduce the dimension of the projective space hence obtained, one can reach the following ([3])

Proposition 1.4 *Every smooth surface can be embedded in* $\mathbb{P}^5$.

This is not true for varieties of higher dimensions, that is for $n > 2$ there exist non-singular abstract varieties of dimension n that cannot be embedded in any projective space. Also, Chow's lemma implies that every variety of arbitrary dimension is birationally equivalent to a projective variety.

Let S be a surface. A Cartier divisor is a formal sum $D = \sum_{i=1}^{n} n_i C_i$ where the C_i are irreducible curves. In general, Cartier divisors are formal sums of irreducible subvarieties of codimension 1. A divisor is said to be effective if each coefficient of the divisor is non-negative, denoted $D \geq 0$. To each function $f \in k(S)$, one can associate a divisor (f), that is the divisor of zeros and poles of f. Two divisors D_1 and D_2 are said to be linearly equivalent, denoted $D_1 \equiv D_2$, if there is a function $f \in k(S)$ such that $D_1 = D_2 + (f)$. Because we have, $(f) + (g) = (fg)$, we have that linear equivalence is an equivalence relation on divisors, and that the operation of addition of the formal sums is well-defined under linear equivalence; hence making the set of linear equivalence classes a group, denoted $\mathrm{Cl}(S)$.

On the other hand, to each divisor D, one can associate an invertible sheaf $\mathcal{O}_S(D)$. Let C be an irreducible curve (taken as a divisor). We define

$$\mathcal{O}_S(C)(U) = \{f \in k(U) : (f) + C_U \geq 0\} \tag{1}$$

where U is taken as a variety and C_U is the divisor of U that is the restriction of C to U. This is an invertible sheaf (i.e. a locally free $\mathcal{O}_S$-module of rank 1). We can define,

$$\mathcal{O}_S(C_1 + C_2) = \mathcal{O}_S(C_1) \otimes \mathcal{O}_S(C_2) \tag{2}$$

and hence extend the association $D \mapsto \mathcal{O}_S(D)$ to all divisors. It is evident that linearly equivalent divisors would give isomorphic invertible sheaves, so this association gives an isomorphism $\mathrm{Cl}(S) \cong \mathrm{Pic}(S)$ between the group of divisors and the group of invertible sheaves with tensor product. Because of the correspondence between divisors and invertible sheaves, we shall sometimes write D for $\mathcal{O}_S(D)$ by an abuse of notation.

Let C_1 and C_2 be two curves of S, let $x \in C_1 \cap C_2$ and let f and g be the local equations of C_1 and C_2 respectively at x, we define

$$(C_1 \,.\, C_2)_x := \dim_{\mathbb{C}} \mathcal{O}_x / < f, g > \tag{3}$$

and since two distinct curves can only intersect at finitely many points, we can set

$$(C_1 \,.\, C_2) = \sum_{x \in C_1 \cap C_2} (C_1 \,.\, C_2)_x \tag{4}$$

Extending this linearly to Cartier divisors we get a symmetric bilinear form on $\mathrm{Pic}(S)$. We shall sometimes denote $(C_1 \,.\, C_2)$ by $C_1.C_2$. Since the form is invariant under linear equivalence, the self intersection of a curve C on S, denoted $C^2 = C.C$ is well-defined through changing one of the entries by the divisor of a function in order to have finite intersection.

This intersection form coincides with the symmetric bilinear form defined on $\mathrm{Pic}(S)$ by

$$(L.L') = \chi(\mathcal{O}_S) - \chi(L^{-1}) - \chi({L'}^{-1}) + \chi(L^{-1} \otimes {L'}^{-1}) \tag{5}$$

where χ is the characteristic $\chi = \sum_i (-1)^i h^i(S, \mathcal{O}_S)$.

A divisor -or a line bundle- E on S is said to be nef if has $E.D \geq 0$ for every effective divisor D on S.

A smooth surface S is said to be minimal if, every birational morphism $S \to S'$, onto a smooth surface S' is an isomorphism. This means that S cannot be blown down; that is it is not the blowup of another smooth surface. This is equivalent to saying that S has no (-1)-curves, that is curves with self-intersection -1, which do not contract to a singularity. If S' is the result of a finite number of blowups of a smooth minimal surface S, S is said to be a minimal model of S'. Because the Neron-Severi group loses rank when we blow down, every surface admits a minimal model. A ruled surface is a surface which is birational to a surface of the form $C \times \mathbb{P}^1$ for a curve C. For whether a surface can have more than one minimal model, we have

Proposition 1.5 *If S is a non-ruled surface, S has a unique minimal model.*

So it makes sense to try to classify minimal surfaces for non-ruled surfaces, in particular for surfaces of general type. We would like to point out that the situation is much more difficult in higher-dimensional varieties.

Let D be a divisor on a surface S. A complete linear system $|D|$ is the set of all effective divisors linearly equivalent to D. To every element of $|D|$, there corresponds a global section of $\mathcal{O}_S(D)$, up to multiplication by a scalar. And to every global section $f \in H^0(S, \mathcal{O}_S(D))$, there corresponds an element of $|D|$. So the space $|D|$ corresponds to the projective space obtained from $H^0(S, \mathcal{O}_S(D))$. A linear system is a subspace of this space; it is at the same time used as its counterpart in $|D|$. We shall usually consider complete linear systems $P = |D|$. The dimension of a linear system P is the dimension of P in $H^0(S, \mathcal{O}_S(D))$. The fixed part of a linear system P is the set of divisors which are present (in the sum) in every element of P. A fixed point (or base point) $x \in S$ of P is a point such that every divisor in P contains a term which contains x.

Let $\phi : S \dashrightarrow \mathbb{P}^n$ be a rational map, by considering the inverse image of the system of hyperplanes in $\mathbb{P}^n$, we obtain a linear system. On the other hand, let P be a linear system on S and P^* be the projective space dual to the projective space P, we get a rational map $\phi : S \dashrightarrow P^*$ by sending $x \in S$ to the hyperplane in P of the divisors passing through x (such a map would be undefined in the base-points of P). These associations give a one to one correspondence between fixed-part-free linear systems of dimension n and rational maps $\phi : S \dashrightarrow \mathbb{P}^n$ such that $\phi(S)$ is not contained in any hyperplane.

A line bundle L, or the associated divisor on a surface S is said to be *very ample* if the linear system $|L|$ induces an embedding of S. An *ample* line bundle is a line bundle L for which there is a positive integer n such that $|nL|$ induces an embedding. Every ample line bundle is nef but the converse is not true in general.

1.2 Main Invariants and Tools

The space of rational 1-forms $\Omega^1_{k(S)/k}$ is the quotient of the vector space over k(S) generated by elements of the form df for $f \in k(S)$, by the relations $d(f+g) = df + dg$, $d(fg) = fdg + gdf$ and $df = 0$ for f constant. We define the space of rational r-forms by $\Omega^r_{k(S)/k} = \bigwedge^r \Omega^1_{k(S)/k}$. An r-form ω is said to be regular at a point p if it can be expressed as

$$\sum f_i dg_{i_1} \wedge ... \wedge dg_{i_r} \tag{6}$$

for $f_i, g_{i_j} \in \mathcal{O}_{S,p}$ at that point. So we can associate, with the space of rational r-forms $\Omega^r_{k(S)/k}$, an invertible sheaf $\mathcal{O}_S(\Omega^r_{k(S)/k})$, such that

$$\mathcal{O}_S(\Omega^r_{k(S)/k})(U) = \{\omega \in \Omega^r_{k(S)/k} : \omega \,\text{regular on}\, U\} \tag{7}$$

For $r = 2$, this sheaf is called the canonical sheaf or the canonical line bundle and the divisor associated to it through the isomorphism between $\mathrm{Cl}(S)$ and $\mathrm{Pic}(S)$ is called the canonical divisor, denoted K.

Some of the invariants we associate to surfaces in order to classify them are

$$\begin{aligned} q &= h^1(S, \mathcal{O}_S) \;\; \text{called the irregularity} \\ p_g &= h^2(S, \mathcal{O}_S) = h^0(S, K) \;\; \text{called the geometric genus} \\ P_n &= h^0(S, nK) \;\; \text{called the plurigenera} \end{aligned} \tag{8}$$

So, $p_g = P_1$. The equality for p_g comes from the Serre Duality Theorem stated below. We also have the self-intersection of the canonical divisor, K^2, the characteristic $\chi = \chi(\mathcal{O}_S)$ where for an invertible sheaf $\mathcal{F}$ on S

$$\chi(\mathcal{F}) = \sum_i (-1)^i h^i(S, \mathcal{F}) \tag{9}$$

and the usual topological invariants: the Betti numbers $b_i = \dim_{\mathcal{R}} H^i(S, \mathbb{R})$ and the topological Euler-Poincaré characteristic $\chi_{top} = \sum_i (-1)^i b_i$. From Hodge theory, we have the equality, $q = b_1/2$. We should note that P_n, p_g,q, χ are all birational invariants, i.e. they are invariant under birational equivalence, whereas b_2, K^2 and hence χ_{top} are not.

Theorem 1.6 *(Serre Duality Theorem) Let S be an algebraic surface, D a divisor on S and K the canonical divisor. Then $h^2(S, K) = 1$ and there is a perfect pairing*

$$H^i(S, D) \otimes H^{2-i}(S, K - D) \to H^2(S, K) \cong \mathbb{C} \tag{10}$$

In particular, we have $\chi(D) = \chi(K - D)$.

We shall also use the Riemann-Roch theorem and classical equality of M. Noether.

Proposition 1.7 *(Riemann-Roch) Let D be a divisor on S, then*

$$\chi(D) = \chi(O_S) + \frac{1}{2}(D^2 - D.K) \tag{11}$$

Theorem 1.8 *(Noether's equality) Let S be an algebraic surface, K its canonical divisor, then*

$$\chi(\mathcal{O}_S) = \frac{1}{12}(K^2 + \chi_{top}(S)) \tag{12}$$

The nth pluricanonical map of a surface S, denoted ϕ_{nK} is the map associated to the linear system of divisors $|nK|$.

Definition 1.9 *The Kodaira dimension of a variety X, denoted $\kappa(X)$ is the maximal dimension of the image $\phi_{nK}(X)$ of X under the pluricanonical maps.*

Equivalently, for surfaces we can set

$$\begin{aligned} \kappa = -\infty &\Leftrightarrow P_n = 0 \,\text{for all}\, n \geq 1 \\ \kappa = 0 &\Leftrightarrow P_n \leq 1 \,\text{for all}\, n \geq 1 \,\text{but sometimes nonzero} \\ \kappa = 1 &\Leftrightarrow P_n \,\text{grows linearly} \\ \kappa = 2 &\Leftrightarrow P_n \,\text{grows quadratically} \end{aligned} \tag{13}$$

Finally, we define one of the main objects of study of surfaces of general type, but we will not make use of it very much.

Definition 1.10 *The canonical ring of a surface S is the graded ring*

$$R(S) = \bigoplus_{n=0}^{\infty} H^0(S, nK) \tag{14}$$

Then it is clear that the Kodaira dimension of S is also given by

$$\kappa(S) = \begin{cases} -\infty, & R(S) \cong \mathbb{C} \\ \mathrm{trdeg}_{\mathbb{C}} R(S) - 1, & \text{otherwise} \end{cases} \tag{15}$$

1.3 The Enriques Classification

The Enriques classification [3] is a broad classification of complex algebraic surfaces based on κ, q and p_g, as follows.

Theorem 1.11 *Every minimal smooth complex algebraic surface falls into one of the following categories according to its Kodaira dimension κ, irregularity q and geometric genus p_g.*

κ	q	p_g	*Name of the surface*
$-\infty$	g	0	*ruled surfaces of genus $g \geq 1$*
	0	0	*rational surfaces*
0	0	0	*Enriques surfaces*
	1	0	*bielliptic surfaces*
	0	1	*K3 surfaces*
	2	1	*Abelian surfaces*
1			*(some) elliptic surfaces*
2			*surfaces of general type*

Ruled surfaces, as mentioned above are surfaces birational to a surface of the form $C \times \mathbb{P}^1$ for a curve C, the g above stands for the genus $g(C) = h^1(C, \mathbb{O}_C)$ of the curve C. A fine classification of ruled surfaces is achieved as follows, a surface S is said to be geometrically ruled over a curve C if there is a surjective morphism $p : S \to C$ with every fibre equal to $\mathbb{P}^1$. Geometrically ruled surfaces are ruled and more importantly, minimal ruled surfaces are all geometrically ruled. To each geometrically ruled surface, there corresponds a projective bundle $\mathbb{P}_C(E)$ for a rank 2 vector bundle E over C unique up to the relation $E' \sim E \otimes L$ for some line bundle L; and vice versa. So, classifying ruled surfaces is equivalent to classifying rank 2 vector bundles under the given relation; which in turn is well understood. *Rational* surfaces are surfaces birational to $\mathbb{P}^2$.

Bielliptic surfaces are surfaces of the form $(E \times F)/G$ for elliptic curves E and F such that G is a finite group of translations of E with an action on F such that $G/F \cong \mathbb{P}^1$.

Elliptic surfaces are surfaces that admit elliptic fibrations, that is for which there exists a proper smooth map $p : S \to C$ onto a curve C such that the general fibre is an elliptic curve. We should point out that bielliptic, hyperelliptic and Enriques surfaces are all elliptic, together with some elliptic $K3$ and elliptic ruled surfaces.

Enriques surfaces are defined as surfaces with $\kappa = 0$, $q = 0$ and $p_g = 0$; which imply that $2K \equiv 0$ for these surfaces. $K3$ surfaces are defined as surfaces with $\kappa = 0$, $q = 0$ and $P_g = 1$; which in turn imply that $K \equiv 0$. The theory of Enriques surfaces and $K3$ surfaces are parallel since Enriques surfaces are quotients of $K3$ by fixed-point-free involutions. $K3$ surfaces enjoy the fact that they are of only one diffeomorphism type, and that they have been found to depend on 19 parameters, i.e. they depend on 19 moduli. [1]

A *complex torus* is a manifold which is a quotient V/Γ of a $\mathbb{C}$-vector space V by a lattice (i.e. $\Gamma \otimes_{\mathbb{Z}} \mathbb{R} \cong V$). An *Abelian variety* is a complex torus that can be embedded in projective space. An *Abelian surface* is a 2-dimensional Abelian variety.

We shall not go into the details of what is known on each of these classes. We can say, however that the fine classification of of all surfaces of special type (i.e. not of general type) has been completed. [1]

The birational geometry of algebraic varieties is generelized by the bimeromorphic geometry of campact complex manifolds. Kodaira later extended the Enriques classification to be a classification of compact complex surfaces. In Kodaira's extension, there are new classes, such as the class VII; as well new surfaces, i.e. generalizations of known classes, such as non-algebraic $K3$ surfaces (which are for this case defined as surfaces with trivial canonical bundle) and complex tori. [1] However, compact complex surfaces of general type are all algebraic. In the rest of this essay, we shall only consider surfaces of general type.

There also is a similar classification of surfaces over fields of characteristic $p > 0$, done by Bombieri and Mumford, with new surfaces arising in characteristics 2 and 3. [5]

2 Surfaces of General Type

2.1 Geography and Examples

A variety S is said to be of general type if it has Kodaira dimension $\kappa(S) = \dim S$. We associate with a minimal surface of general type, two main invariants: the Chern invariants $c_1^2 = K^2$ and $c_2 = \chi_{top}$. Even though these are not birational invariants, there is no problem of well-definedness since we are only trying to classify the minimal surfaces in view of the fact that non-ruled surfaces have unique minimal models. The chern invariants of a minimal surface of general type determine its plurigenera (see Proposition 2.7 below). Some authors prefer using K^2 and $\chi(S) = (c_1^2 + c_2)/12$ in view of Noether's equality (Theorem 1.8) and refer to them as the Chern invariants but we shall use K^2 and χ_{top}.

We shall give a few examples of surfaces of general type.

Example 2.1 The simplest examples of surfaces of general type are products of the form $C_1 \times C_2$ for curves C_1 and C_2 each of genus at least 2. The fact that these are surfaces of general type follows from the following lemma, which describes the Kodaira dimension of a product of curves.

Lemma 2.2 *If C_1 and C_2 are smooth algebraic curves, then*

$$P_n(C_1 \times C_2) = P_n(C_1).P_n(C_2).$$

Proof: Let $\pi_1 : C_1 \times C_2 \to C_1$ and $\pi_2 : C_1 \times C_2 \to C_2$ be the projections. Then, for $n \in \mathbb{N}$, it is clear that

$$K_{C_1 \times C_2} = \pi_1^* K_{C_1} + \pi_2^* K_{C_2} \tag{16}$$

So we have

$$H^0(C_1 \times C_2, nK_{C_1 \times C_2}) = H^0(C_1, nK_{C_1}) \otimes H^0(C_2, nK_{C_2}) \tag{17}$$

Thus $P_n(C_1 \times C_2) = P_n(C_1).P_n(C_2)$. □

So, in particular we have, for the product of two smooth curves C_1 and C_2

$$\kappa(C_1 \times C_2) = \begin{cases} -\infty, & C_1 \text{ or } C_2 \text{ is rational} \\ 0, & C_1 \text{ and } C_2 \text{ are both elliptic} \\ 1, & C_1 \text{ is elliptic and } g(C_2) \geq 2 \\ 2, & \text{otherwise} \end{cases} \tag{18}$$

Example 2.3 *(Complete intersections)* Denote by $S_{d_1,\ldots,d_n}$ some surface that is a complete intersection of n hypersurfaces of degrees $d_1, d_2, \ldots, d_n$ in $\mathbb{P}^{n+2}$. The canonical divisor of such a surface is $K = ((\sum_i d_i) - n + 3)H$. So, $S_{d_1,\ldots,d_n}$ is of general type unless $(\sum_i d_i) - n - 3 \leq 0$; which is the case only for S_2, S_3, $S_{2,2}$ for which the resulting surfaces are rational; and for S_4, $S_{2,3}$ and $S_{2,2,2}$ for which the resulting surfaces have Kodaira dimension 0 and fall into the category of K3 surfaces.

Example 2.4 *(The Godeaux Surface)* Let S' be the surface in $\mathbb{P}^3$ given by the homogeneous equation $x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0$ and let the group $G = \{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$ where ζ is a primitive 5th root of unity act on S' by

$$\zeta \cdot (x_1, x_2, x_3, x_4) = (x_1, \zeta x_2, \zeta^2 x_3, \zeta^3 x_4) \tag{19}$$

Let $S = S'/G$ and let $p : S' \to S$ be the projection. Then S is an algebraic surface, in fact a projective algebraic surface, by virtue of the finiteness of the map p and the fact that if on a surface, there exists a line bundle L such that $c_1(L) > 0$, then the surface is projective. Moreover, S is smooth since the G-action S' has no fixed points. Furthermore, S is of general type because $P_n = 1 + \frac{1}{2}(n^2 - n)$. *Godeaux's construction inspired Serre to show that every finite group G is the fundamental group of an algebraic surface, which could perhaps give one the impression that the classification of algebraic surfaces should be harder than the classification of finite groups.*

The Chern invariants of a surface of general type are restricted to a clear-cut region of the plane. (See Figure 1 below)

Theorem 2.5 *Let S be a minimal surface of general type. Then,*

(i) $c_1^2 + c_2 \cong 0 (\mathrm{mod}\ 12)$

(ii) $c_1^2 > 0$

(iii) $c_2 > 0$

(iv) (Bogomolov-Miyaoka-Yau inequality) $c_1^2 \leq 3c_2$

(v) $5c_1^2 - c_2 + 36 > 0$ *when* c_1^2 *is even*

(vi) $5c_1^2 - c_2 + 30 > 0$ *when* c_1^2 *is odd*

Proof: (i) is clear from Noether's equality (Theorem 1.8).

(ii) Let H be a smooth hyperplane section of S in some embedding of S into projective space. Consider the exact sequence,

$$0 \to \mathcal{O}_S(nK - H) \to \mathcal{O}_S(nK) \to \mathcal{O}_H(nK) \to 0 \tag{20}$$

and the exact cohomology sequence obtained through it. Because, $h^0\mathcal{O}_S(nK)$ grows quadratically, and $h^0\mathcal{O}_H(nK)$ grows linearly, there is an n_0 such that the divisor $n_0K - H$ is linearly equivalent to an effective divisor R. Because K is nef, we have:

$$n_0^2K^2 = n_0K(H + R) = n_0K.H + n_0K.R \geq n_0K.H = (H^2 + H.R) \geq H^2 > 0$$

Sketch of proof for (iii) Assume that there exists a surface S of general type with $\chi_{top}(S) < 0$. Then one can show, using results from Hodge theory, that there is a surface S' such that there is a surjective étale morphism $p : S' \to S$, – i.e. a morphism such that the induced map between the tangent spaces at each point is an isomorphism – and such that $\chi_{top}(S') < 0$ and $p_g(S') \leq 2q(S') - 4$. But this numerical condition implies that there are, on S' two one-forms ω_1, $\omega_2 \in \Omega^1_{S'}$ which are linearly independent such that $\omega_1 \wedge \omega_2 = 0$. And this property implies that S' must admit a surjective morphism $\pi : S' \to B$ onto a curve B of genus at least two, with connected fibres, and say, generic fibre F. Then using the fact that in the case of the existence of such a morphism, $\chi_{top}(S') \geq \chi_{top}(B).\chi_{top}(F)$; one reaches that $\chi_{top}(S') \geq 0$, a contradiction. [3]

Sketch of proof for (iv): Assume $a = \frac{c_2}{c_1^2} < \frac{1}{3}$. Let $b = \frac{1}{4}(1 - 3a)$ and let n be a positive integer such that $n(a + b) \in \mathbb{Z}$. Consider the line bundle,

$$V_n = S^n\Omega^1_S \otimes \mathcal{O}_S(-n(a + b)K) \tag{21}$$

One can show that $h^2(S, V_n) = h^0(S, V_n) = 0$ for sufficiently large n, which implies that $\chi(V_n) \leq 0$. But on the other hand, $\chi(V_n)$ can be shown to be a polynomial of degree 3 using the Riemann-Roch theorem. [1], [16]

(v) *and* (vi): Since by definition, $\chi(S) = 1 - q(S) + p_g(S)$, one obtains, from Noether's inequality [1]:

$$p_g \leq \frac{1}{2}c_1^2 + 2 \tag{22}$$

that

$$\chi(S) \leq 1 + p_g \leq \frac{1}{2}c_1^2 + 3 \tag{23}$$

If c_1^2 is even, we can translate this directly using Noether's formula, to obtain

$$5c_1^2 + c_2 + 36 \geq 0 \tag{24}$$

If c_1^2 is odd, we can remove the extra $\frac{1}{2}$ and translate to obtain a sharper

$$5c_1^2 + c_2 + 30 \geq 0 \tag{25}$$

□

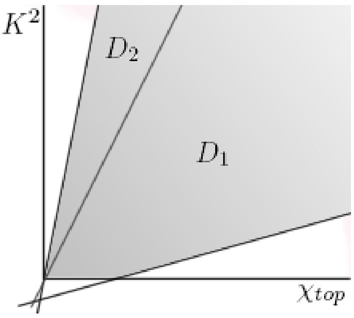

Figure 1: The region in which Chern invariants of minimal surfaces of general type live

The possible values of the Chern invariants according to the above theorem are the regions D_1 and D_2 of the diagram above and their positive c_1^2, c_2 - boundaries. It is thought that surfaces in the region D_2 are much harder to construct. The geography problem, as coined by Persson is the problem of finding which admissible values of the Chern invariants, i.e. which values satisfying the above theorem, can occur. A considerable amount of effort has gone into constructing surfaces with given Chern invariants and the result can be summed up, as done in [1] as follows,

Theorem 2.6 *(Geography theorem) For each pair* (m, n) *satisfying the results of Theorem 2.5, there is a minimal surface of general type with* $c_1^2 = m$ *and* $c_2 = n$*; except maybe for the pairs on the finitely many lines* $m - 3m + 4k = 0$ *with* $0 \leq k \leq 347$.

Investigations on how other properties of surfaces of general type relate to the restictions or necessities on the invariants is also considered to be part of surface geography. The supreme example of this is Yau's theorem, which states that surfaces on the Bogomolov-Miyaoka-Yau line, i.e. surfaces with $K^2 = 3\chi_{top}$ are quotients of the unit ball [21]. Surfaces close to the Noether lines (the lines in Theorem 2.5-(v),(vi)), simply connectedness and having $q = p_g = 0$ also have been studied in the context of geography. [18], [1]

For minimal surfaces of general type, the chern invariants determine the plurigenera.

Proposition 2.7 *Let S be a minimal surface of general type, then for $n \geq 2$*

$$P_n(S) = \chi(\mathcal{O}_S) + \frac{1}{2}(n^2 - n)K^2 \tag{26}$$

Proof: by the Riemann-Roch theorem, we have

$$h^0(S, nK) - h^1(S, nK) + h^2(S, nK) = \chi(\mathcal{O}_S) + \frac{1}{2}(n^2 - n)K^2 \tag{27}$$

So, it suffices to show that $h^1(S, nK) = h^2(S, nK) = 0$. By Serre Duality, we have

$$h^2(S, nK) = h^0(S, -(n-1)K)h^1(S, nK) = h^1(S, -(n-1)K) \tag{28}$$

But then, Mumford's vanishing theorem, which states that 0th and 1st cohomologies of the duals (inverses) of nef line bundles with positive first chern class vanish, which is true for our case by theorem 2.5 and by the assumption that $n \geq 2$; completes the proof. □

2.2 The Pluricanonical Maps

Let S be a minimal surface of general type and let $\phi_{nK} : S \longrightarrow \mathbb{P}^{h^0(nK)-1}$ be the nth pluricanonical map. The most important invariant under consideration being the Kodaira dimension which can also be described as the maximal dimension of the images of S under the pluricanonical maps; one can ask what kind of information one can obtain from the image of S under these maps. Bombieri has proven: [6]

Theorem 2.8 *If S is a minimal surface of general type, then ϕ_{nK} is a birational morphism onto its image for all $n \geq 5$.*

An example of a surface not birational to its fourth canonical image is the Godeaux surface discussed in Example 2.4. To prove Theorem 2.8, we shall use Reider's method. [19]

Theorem 2.9 *Let S be a projective surface and L be a nef divisor on S.*

- *If $L^2 \geq 5$ and P is a basepoint of the linear system $|K + L|$, then there exists a divisor D on S with $P \in D$ such that:*

  a) $D.L = 1$ *and* $D^2 = 0$.

  or b) $D.L = 0$ *and* $D^2 = -1$

- *If $L^2 \geq 9$ and P_1 and P_2 are points of S that are not basepoints of the linear system $|K + L|$, then there is a divisor D, passing through both P_1 and P_2, such that:*

    a) $D.L = 0$ *and* $D^2 = -2$ *or* -1.

    or b) $D.L = 1$ *and* $D^2 = -1$ *or* 0.

    or c) $D.L = 2$ *and* $D^2 = 0$.

    or d) only when $L^2 = 9$, $c_1(L)$ *(considering L as a line bundle) is cohomologous to* $3D$ *modulo torsion, and* $D^2 = 1$

For a proof of this theorem, which is based on Bogomolov unstability of vector bundles, we refer the reader to [19] and [1]. We need the following lemma for Theorem 2.8.

Lemma 2.10 *On a minimal surface of general type, there are finitely many curves with self-intersection* -2.

Proof: Since each rational homology class, in $H_2(S, \mathbb{Q})$ can contain only one (-2)-curve and the Neron-Severi group has finite rank, there can only be finitely many such curves. $\square$

Proof of Theorem 2.8: Let $n \geq 5$ and $L = (n-1)K$. Since S is minimal of general type K is nef, so L is nef. Also since $L^2 = (n-1)K^2 \geq 16$, we can use Reider's theorem. Assume ϕ_{nK} is not defined everywhere; which is equivalent to saying that the linear system $|nK| = |(n-1)K + K|$ has a basepoint P. By Reider's theorem, there must be a curve D going through P with $D.(n-1)K = 1$ (and $D^2 = 0$), which is not possible since $n \geq 5$; or $D.L = 0$ and $D^2 = -1$, which is also impossible since, $D^2 - K.D \cong 0 (\text{mod } 2)$ by the Riemann-Roch theorem. So the linear system $|nK|$ cannot have a basepoint and hence the map ϕ_{nK} is defined everywhere (i.e. it is a morphism). Now assume that there were two points on S_0, say P and Q that were not separated by ϕ_{nK}, or equivalently were not separated by the linear system $|nK|$. These points cannot be basepoints of $|nK|$ by the above argument, so we can again apply Reider's theorem. Again, $D.(n-1)K$ cannot be 1 or 2 because $n - 1 \geq 4$. And again by the Riemann-Roch theorem, we cannot have $D.L = 0$ and $D^2 = -1$. So the only possible case for D is to be a (-2)-curve with $K.D = 0$. So, there is a (-2)-curve going through P and Q. So, only on the (-2)-curves can ϕ_{nK} not be injective. But since we only have finitely many (-2)-curves on S, the subset of S consisting of the elements that are not on the (-2)-curves is an open dense subset of S. Since ϕ_{nK} is injective on this subset, the image of ϕ_{nK} is two-dimensional and ϕ_{nK}

is birational onto its image. □

In fact, one can be more descriptive. Since on a minimal surface S of general type, there are finitely many (-2)-curves. We can consider the union of these curves and its connected components, say C_i, which can be blown down to give a possibly singular surface (so the curves are not 'exceptional' in the way we defined the term, because blowing them down does not result in a smooth surface). Let S_0 be the singular surface obtained through blowing down the C_i. Since ϕ_{nK} is defined everywhere, ϕ_{nK} induces a map $k_n : S_0 \to \mathbb{P}^{h^0(nK)-1}$. k_n is an embedding for $n \geq 5$. One can still ask which conditions are required in order to have ϕ_{nK} a morphism, birational, or k_n an embedding, for $n < 5$. We have: [1]

Theorem 2.11 *Let S be a minimal surface of general type:*

- *(i) k_4 is an embedding if $c_1^2 \geq 2$*
- *(ii) k_3 is a morphism for $c_1^2 \geq 2$ and an embedding for $c_1^2 \geq 3$*
- *(iii) k_2 is a morphism for $c_1^2 \geq 5$. If $c_1^2 \geq 10$, then k_2 is birational if and only if S does not admit a surjective morphism onto a curve, with a curve of genus 2 as the general fibre.*

Remark 2.12 Let S be a minimal surface of general type, as a corollary to Theorem 2.8, once can show that the canonical ring $R(S)$ is finitely generated and noetherian. So, one can set

$$\mathcal{R}(S) = \mathrm{Proj}(R(S)) \tag{29}$$

and obtain a surface. $\mathcal{R}(S)$ is called the abstract canonical model of S. Also, the surface $\mathcal{R}^n(S) = \mathrm{Proj}(R^n(S))$ where

$$R^n(S) = \bigoplus_m H^0(S, mnK) \tag{30}$$

is isomorphic to the nth canonical image. It is then clear that two surfaces S and S' of general type are isomorphic if and only if $R(S) \cong R(S')$. We refer the reader to [4] and [10] for more about the canonical models of surfaces of general type.

2.3 Moduli Spaces of Surfaces With Given Chern Invariants

When we wish to classify the elements of a certain set of objects with respect to a given equivalence relation, we may want to see if the set of equivalence classes can have a structure similar to the objects we are trying to classify. For example, curves of genus g and stable vector bundles on curves vary over algebraic families. The idea is the one of a moduli space.

Let S be the set of isomorphism classes of the objects we wish to classify up to another equivalence relation, say $\sim$ (e.g. birational equivalence). A coarse moduli variety will be, for us a variety M, with a map $m : S \to M$ such that

i) $m(S_1) = m(S_2)$ if and only if $S_1 \sim S_2$

ii) For every deformation in S, i.e. for every map $f : Y \to T$ between two varieties Y and T such that the fibers of f are all in S, the map $g : T \to M$ given by $g(x) = m(f^{-1}(x))$ is an algebraic morphism.

We have the following result of Gieseker [13].

Theorem 2.13 *(Gieseker) There exists a quasiprojective moduli variety $\mathcal{M}_{K^2,\chi_{top}}$ parameterizing the surfaces of general type with given given Chern invariants K^2 and χ_{top}.*

We shall call $\mathcal{M}_{K^2,\chi_{top}}$ the Gieseker moduli space (or scheme, or variety) for K^2 and χ_{top}. The proof uses the Hilbert Scheme technique and methods of Mumford's Geometric Invariant Theory [17]. We will not give a full proof of this fact, but we shall illustrate the the main idea.

Let V be a vector space and G be a linear algebraic group acting on $\mathbb{P}(V)$, where $\mathbb{P}(V)$ is the set of hyperplanes in V. A point $x \in \mathbb{P}(V)$ is said to be stable if

(i) the stabilizer of x under the action of G is finite

(ii) for any point $x_0 \in V$ corresponding to $x \in \mathbb{P}(V)$ the orbit $G \cdot x$ is a closed subset of V^*

In GIT, Mumford has shown that the stable points of V under the action of a reductive algebraic group G form an open subset, say U. Moreover, there are invariant sections (under the group action), u_0, u_1, ..., u_N of the sheaf $\mathcal{O}_{\mathbb{P}(V)}(p)$. so that the map $I : U/G \to \mathbb{P}^N$ is well defined and injective. And also that the image of I is locally closed in $\mathbb{P}^N$.

Now, Let X be a surface of general type. Let P_n denote the nth plurigenerus as usual. By a choice of basis, say X_1, X_2, ... X_{P_n} of $H^0(X, \mathcal{O}(nK))$, one has an isomorphism $\phi : H^0(X, \mathcal{O}(nK)) \to \mathbb{C}^{P_n}$. By Theorem 2.8, we can take $n \geq 5$ so that the induced map f_n would be everywhere defined. Denote the image $f_n(X)$ by (X_n, ϕ). This image has at most rational double points because it is obtained by a contraction of the (-2)-curves on the surface. Let m be a large integer. Then there is a natural map

$$\mathrm{Sym}^m(H^0(X, \mathcal{O}(nK))) \to H^0(X, \mathcal{O}(mnK)) \tag{31}$$

that is surjective; where $\mathrm{Sym}(V)$, for a module V is the symmetric algebra of V, that is $\mathrm{Sym}(V)$ is the quotient of the tensor algebra by the ideal $\langle x \otimes y - y \otimes x \,|\, x, y \in V \rangle$. So, with the isomorphism ϕ, we get a surjective map:

$$\psi_{(X_n,\phi)} : \mathrm{Sym}^m(\mathbb{C}^{P_n} \to H^0(X, \mathcal{O}(mnK))) \tag{32}$$

Then clearly, the kernel of this map is the set of homogeneous polynomials of degree m on X_1, X_2, ... X_{P_n}, which vanish on (X_n, ϕ). For m sufficiently large, these equations would determine (X_n, ϕ) and hence determine X. But this kernel, on the other hand is determined by the map

$$\psi_{(X_n,\phi)} : \bigwedge^r \mathrm{Sym}^m(\mathbb{C}^{P_n}) \to \bigwedge^r H^0(X, \mathcal{O}(mnK)) \tag{33}$$

Thus, each pair (surface X, choice of basis) gives us a well-defined element of the set $\mathbb{P}(\bigwedge^r \mathrm{Sym}^m(\mathbb{C}^{P_n}))$, let us denote this element by $\beta(X, \phi)$. $G = SL(P_n, \mathbb{C})$ acts naturally and freely on the set $\mathbb{P}(\bigwedge^r \mathrm{Sym}^m(\mathbb{C}^{P_n}))$ and the orbit $G \cdot \beta(X, \phi)$ of $\beta(X, \phi)$ consists of elements of the form $\beta(X, \phi')$ for other isomorphisms $\phi' : H^0(X, \mathcal{O}(NK)) \to \mathbb{C}^{P_n}$. Thus, the hence well-defined map $m_0 : S \to Orb(G, \mathbb{P}(\bigwedge^r \mathrm{Sym}^m(\mathbb{C}^{P_n})))$ is one-to-one for surfaces that are not birational to each other.

We can now apply Mumford's results discussed above to the vector space $V = \mathbb{P}(\bigwedge^r \mathrm{Sym}^m(\mathbb{C}^{P_n}))$. Assume that $\beta(X, \phi)$ are stable. Then there exist some invariant sections u_0, u_1, ..., u_N of the sheaf $\mathcal{O}_{\mathbb{P}(V)}(p)$, such that for sufficiently large p, m and n, some of the $u_i(\beta(X, \phi))$ are non-zero. So we can successfully define the map $m : S \to \mathbb{P}^N$

$$m(\beta(X, \phi)) := (u_0(\beta(X, \phi)), ..., u_N(\beta(X, \phi))) \tag{34}$$

So, it is sufficient to show that $\beta(X_{n,\phi})$ are stable points in order to construct m so that it satisfies the first property for a coarse moduli space. Gieseker does this by using Mumford's numerical criterion for stability and hard combinatorial estimates.

In this setting, the geography problem asks whether the moduli space $\mathcal{M}_{K^2,\chi_{top}}$ is empty for some admissible values of the Chern invariants. Which has not been settled yet as discussed above.

Even though Gieseker's theorem has been an important development, there still is very little that is known about the geometry of these moduli spaces. Some simple questions to ask would be the following: How many irreducible or connected components does the moduli space $\mathcal{M}_{K^2,\chi_{top}}$ have? Note that there must be finitely many irreducible or connected components since $\mathcal{M}_{K^2,\chi_{top}}$ is quasi-projective. Is the moduli space of pure dimension? (i.e. are the components all of the same dimension). We shall end this expository essay with some results on the geometry of these moduli schemes.

Definition 2.14 *Two smooth surfaces S and S' are said to be deformation equivalent if there exists a morphism $p : T \to B$ of algebraic varieties, such that the fibers of p are all smooth and such that S and S' are both fibers of p.*

In fact, one can define deformations of surfaces without the requirement that fibers are algebraic, but such deformations preserve the class of surfaces in the Enriques-Kodaira classification (and also preserve minimality for surfaces with $\kappa \geq 0$), so the definitions are the same for the purpose of studying surfaces of general type. If two surfaces are deformation equivalent, then they are diffeomorphic. The converse however, is not true. [12] But more importantly, the following is a clear result of the definitions

Proposition 2.15 *If S and S' are minimal surfaces of general type, then S and S' are in the same connected component of the Gieseker moduli scheme if and only if they are deformation equivalent.*

The following theorem of Catanese is a partial answer to the first and an answer to the second questions about the geometry of the Gieseker moduli scheme.

Theorem 2.16 *For every positive integer c, there is a pair (a, b) such that the Gieseker moduli scheme for surfaces with $K^2 = a$ and $\chi_{top} = b$ has at least c irreducible components all of which have different dimensions.*

Catanese proves this in [8] by constructing suitable bidouble covers of the quartic, that is, Galois covers of $\mathbb{P}^1 \times \mathbb{P}^1$ with group $\mathbb{Z}_2 \times \mathbb{Z}_2$. He also shows, using Freedman's result concerning the classification of four-manifolds which states that if S and S' are compact, oriented, simply connected, smooth four-manifolds with the same intersection form on $H^2(S, Z)$ and $H^2(S', Z)$ respectively, then they are homeomorphic (with the same orientation); that these surfaces are all orientedly homeomorphic. Later, using the same construction method, Catanese shows that one can also construct k different surfaces, all in different connected components of the moduli scheme. [9]

Let us denote by $m(S)$, the number dimensions of the irreducible component of the Gieseker moduli scheme on which the point corresponding to S is. $m(S)$ is called the number of moduli of S. We have, for a minimal surface S of general type, the following rough estimate on $m(S)$. [8]

$$\frac{10}{12}K^2 - \frac{14}{12}\chi_{top} \leq m \leq \frac{10}{12}K^2 + \frac{46}{12}\chi_{top} + 108 \tag{35}$$

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