Math 104 Exam 2, Fall 2004
Solutions

Part 1: Multiple Choice

1. Evaluate $\displaystyle\int_3^6 \frac{dx}{x\sqrt{x^2-9}}$.

a) $\pi/2$	b) $\pi/9$	c) 0
d) $\ln\sqrt{27}$	e) $\ln(\sqrt{2}+1)$	f) divergent

We can either use an integration formula or (inverse) trigonometric substitution. If we use a formula, we use

$$\int \frac{dx}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C\ .$$

So we have $a = 3$ in this case. Note that this is an improper integral, since the integrand is discontinuous at $x = 3$. We get

$$\begin{aligned}
\int_3^6 \frac{dx}{x\sqrt{x^2-9}} &= \lim_{s\to3^+}\int_s^6 \frac{dx}{x\sqrt{x^2-9}} = \lim_{s\to3^+}\frac{1}{3}\sec^{-1}\left|\frac{x}{3}\right|\,\Big|_s^6 \\
&= \lim_{s\to3^+}\frac{1}{3}\left(\sec^{-1}(2)-\sec^{-1}\left(\frac{s}{3}\right)\right) = \frac{1}{3}\left(\sec^{-1}(2)-\sec^{-1}(1)\right) \\
&= \frac{1}{3}\left(\frac{\pi}{3}-0\right) = \frac{\pi}{9}
\end{aligned}$$

which is answer (b).

2. Find the absolute minimum value of θ when

$$\theta = \tan^{-1}(x-2) - \tan^{-1}(x), \qquad x \in (-\infty, \infty)\ .$$

a) π	b) $\pi/2$	c) $\pi/4$	d) $\pi/3$
e) 0	f) $-\pi/4$	g) $-\pi$	h) $-\pi/2$

The derivative of θ is

$$\frac{d\theta}{dx} = \frac{1}{1+(x-2)^2} - \frac{1}{1+x^2} = \frac{1}{5-4x+x^2} - \frac{1}{1+x^2}\ .$$

The derivative is defined for all x, so the critical numbers occur when $d\theta/dx = 0$.

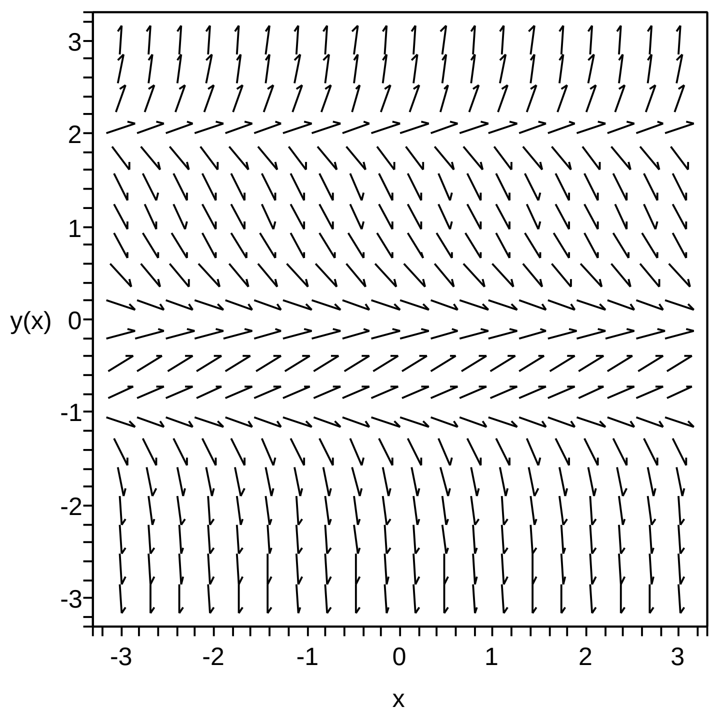
$$\begin{aligned} \frac{d\theta}{dx} &= 0 \\ \Longleftrightarrow \quad \frac{1}{5-4x+x^2} &= \frac{1}{1+x^2} \\ \Longleftrightarrow \quad 1+x^2 &= 5-4x+x^2 \\ \Longleftrightarrow \quad 4x &= 4 \\ \Longleftrightarrow \quad x &= 1\ . \end{aligned}$$

We have $\theta(1) = \tan^{-1}(-1) - \tan^{-1}(1) = -2\ \tan^{-1}(1) = -2\,\pi/4 = -\pi/2$.
To check that this is really an absolute minimum, let us see what happens to $\theta(x)$ when $x \to \pm\infty$.

$$\begin{aligned} \lim_{x\to+\infty} \tan^{-1}(x-2) - \tan^{-1}(x) &= \pi/2 - \pi/2 \ = \ 0 \\ \lim_{x\to-\infty} \tan^{-1}(x-2) - \tan^{-1}(x) &= -\pi/2 + \pi/2 \ = \ 0 \end{aligned}$$

Hence $\theta(1) = -\pi/2$ is the absolute minimum value of θ, which is answer (h).

3. Which of the differential equations has the following slope field?



a) $y' = (y^2 - 1)(y + 4)$

b) $y' = xy$

c) $y' = y^2 - 2y + 1$

d) $y' = x^2 - y^2$

e) $y' = y(y^2 - y - 2)$

f) $y' = \sqrt{x + y}$

We can see from the slope field that the slopes are the same along each horizontal line. This means that y' only depends on y, i.e. $y' = f(y)$. Hence the differential equation is either (a),(c), or (e). We can also see from the slope field that the sign of y' changes at $y = -1$, $y = 0$, and $y = 2$.
a) can be rewritten as $y' = (y + 4)(y + 1)(y - 1)$, so y' changes sign at $y = -4, -1, 1$.
c) can be rewritten as $y' = (y - 1)^2$, so always $y' \geq 0$.
e) can be rewritten as $y' = (y + 1)y(y - 2)$, so y' changes sign at $y = -1, 0, 2$.
This means the answer is (e).

4. Evaluate $\int_0^{\pi/3} \tan^3 x \sec^4 x \, dx$.

a) $27/4$ b) $9/4$ c) $\sqrt{2}/2 - 1$ d) 0

e) $\sqrt{3}/2 - 2$ f) $1/\sqrt{3}$ g) $2\sqrt{2} - 2$ h) ∞

We use that $\sec^2 x = \tan^2 x + 1$ to reduce the exponent of $\sec x$ to 2. Thus we obtain

$$\begin{aligned}
\int_0^{\pi/3} \tan^3 x \sec^4 x \, dx &= \int_0^{\pi/3} \tan^3 x \,(\tan^2 x + 1) \sec^2 x \, dx \\
& \quad \text{substitution: } u = \tan x,\ du = \sec^2 x \, dx \\
& \quad \text{new limits: } x = 0 \Rightarrow u = 0, \text{ and } x = \pi/3 \Rightarrow u = \sqrt{3} \\
&= \int_0^{\sqrt{3}} u^3(u^2+1) \, du \\
&= \left(\frac{1}{6}u^6 + \frac{1}{4}u^4\right) \Big|_0^{\sqrt{3}} \\
&= \frac{27}{6} + \frac{9}{4} = \frac{27}{4}
\end{aligned}$$

which is answer (a).

Another way:

We can also use $\tan^2 x = \sec^2 x - 1$ to reduce the exponent of $\tan x$ to 1. Then we obtain

$$\begin{aligned}
\int_0^{\pi/3} \tan^3 x \sec^4 x \, dx &= \int_0^{\pi/3} \tan x \,(\sec^2 x - 1) \sec^4 x \, dx \\
& \quad \text{substitution: } u = \sec x,\ du = \sec x \tan x \, dx \\
& \quad \text{new limits: } x = 0 \Rightarrow u = 1, \text{ and } x = \pi/3 \Rightarrow u = 2 \\
&= \int_0^2 (u^2-1)u^3 \, du \\
&= \left(\frac{1}{6}u^6 - \frac{1}{4}u^4\right) \Big|_1^2 \\
&= \frac{64}{6} - \frac{16}{4} - \left(\frac{1}{6} - \frac{1}{4}\right) = \frac{27}{4}
\end{aligned}$$

which is again answer (a).

5. Evaluate $\int_0^1 \frac{x+1}{x^2+x-6}\,dx$.

a) $\frac{4}{3}$ b) $\frac{2\ln 5 - 3\ln 4}{3}$ c) $\frac{\ln 6 - \ln 5}{5}$ d) 1

e) $\frac{11}{6}$ f) $\frac{\ln 2 + \ln 3}{6}$ g) $\frac{\ln 2 - 2\ln 3}{5}$ h) 0

We expand the integrand into partial fractions:

$$\frac{x+1}{x^2+x-6} = \frac{x+1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}\ .$$

To find A and B we multiply through by $(x+3)(x-2)$ to get

$$x+1 = A(x-2) + B(x+3)\ .$$

Now we plug in the zeros of $(x+3)(x-2)$:

$$\begin{array}{rcrcl} x=-3: & -2 & = & -5A & \Rightarrow\ A = 2/5, \\ x=\ \ 2: & 3 & = & 5B & \Rightarrow\ B = 3/5\ . \end{array}$$

We obtain

$$\begin{aligned} \int_0^1 \frac{x+1}{x^2+x-6}\,dx &= \int_0^1 \left(\frac{2}{5}\,\frac{1}{x+3} + \frac{3}{5}\,\frac{1}{x-2}\right)\,dx \\ &= \left(\frac{2}{5}\,\ln|x+3| + \frac{3}{5}\,\ln|x-2|\right)\Big|_0^1 \\ &= \frac{2}{5}\,\ln 4 + \frac{3}{5}\,\ln 1 - \left(\frac{2}{5}\,\ln 3 + \frac{3}{5}\,\ln 2\right) \\ &= \frac{4}{5}\,\ln 2 - \frac{2}{5}\,\ln 3 - \frac{3}{5}\,\ln 2 \\ &= \frac{\ln 2 - 2\ln 3}{5} \end{aligned}$$

which is answer (g).

Part 2: Free Response

1. (10 points) Find the limit

$$\lim_{x\to 0^+}\left(1+x+x^2\right)^{1/x} .$$

This is an indeterminate form, namely 1^∞. Hence we first evaluate

$$\lim_{x\to 0^+}\ln\left((1+x+x^2)^{1/x}\right) .$$

We get

$$\begin{aligned}
\lim_{x\to 0^+}\ln\left((1+x+x^2)^{1/x}\right) &= \lim_{x\to 0^+}\frac{1}{x}\ln(1+x+x^2) \\
&= \lim_{x\to 0^+}\frac{\ln(1+x+x^2)}{x} \\
&\quad \text{This is the indeterminate form } \frac{0}{0}, \\
&\quad \text{so we can use l'Hôpital.} \\
&= \lim_{x\to 0^+}\frac{\frac{1+2x}{1+x+x^2}}{1} \\
&= \lim_{x\to 0^+}\frac{1+2x}{1+x+x^2} \;=\; 1 .
\end{aligned}$$

So $\lim_{x\to 0^+}\ln\left((1+x+x^2)^{1/x}\right)=1$, which means that

$$\lim_{x\to 0^+}\left(1+x+x^2\right)^{1/x}=e .$$

2. (10 points) Solve the initial value problem

$$(x+1)\,\frac{dy}{dx} + 2y = x, \quad x > -1, \quad y(0) = 1 \; .$$

We first bring the differential equation into standard form:

$$\frac{dy}{dx} + \frac{2}{x+1}\,y \;=\; \frac{x}{x+1}\; .$$

Hence $P(x) = \dfrac{2}{x+1}$, and $Q(x) = \dfrac{x}{x+1}$.

The integrating factor is now

$$v(x) = e^{\int P(x)\,dx} = e^{2\,\ln|x+1|} = (x+1)^2 \; .$$

Thus the solution of the differential equation is

$$y \;=\; \frac{\int v(x)\,Q(x)\;dx + C}{v(x)}$$

$$y \;=\; \frac{\int x(x+1)\;dx + C}{(x+1)^2}$$

$$y \;=\; \frac{x^3/3 + x^2/2 + C}{(x+1)^2}$$

Now we use the initial condition $y(0) = 1$.

$$1 = y(0) = \frac{0+C}{1^2} = C \; ,$$

so $C = 1$. Thus the solution of the initial value problem is

$$y(x) = \frac{x^3/3 + x^2/2 + 1}{(x+1)^2} \; .$$

3. (15 points) Evaluate the integral

$$\int e^{-x} \sin(3x)\, dx.$$

We use integration by parts twice. Using $\begin{array}{rlrl} u = & e^{-x}, & dv = & \sin(3x)\, dx, \\ du = & -e^{-x}\, dx, & v = & -(1/3)\cos(3x), \end{array}$

we obtain

$$\int e^{-x} \sin(3x)\, dx = -\frac{1}{3} e^{-x} \cos(3x) - \int \frac{1}{3} e^{-x} \cos(3x)\, dx\ .$$

Now we use $\begin{array}{rlrl} u = & e^{-x}, & dv = & (1/3)\cos(3x)\, dx, \\ du = & -e^{-x}\, dx, & v = & (1/9)\sin(3x). \end{array}$

We obtain

$$\begin{aligned} \int e^{-x} \sin(3x)\, dx &= -\frac{1}{3} e^{-x} \cos(3x) - \left(\frac{1}{9} e^{-x} \sin(3x) + \int \frac{1}{9} e^{-x} \sin(3x)\, dx\right) \\ \int e^{-x} \sin(3x)\, dx &= -\frac{1}{3} e^{-x} \cos(3x) - \frac{1}{9} e^{-x} \sin(3x) - \int \frac{1}{9} e^{-x} \sin(3x)\, dx \end{aligned}$$

Now we solve for $\int e^{-x} \sin(3x)\, dx$. We obtain

$$\begin{aligned} \frac{10}{9} \int e^{-x} \sin(3x)\, dx &= -\frac{1}{3} e^{-x} \cos(3x) - \frac{1}{9} e^{-x} \sin(3x) + C \\ &= -e^{-x} \left(\frac{1}{3} \cos(3x) + \frac{1}{9} \sin(3x)\right) + C\ . \end{aligned}$$

Thus

$$\begin{aligned} \int e^{-x} \sin(3x)\, dx &= -\frac{9}{10} e^{-x} \left(\frac{1}{3} \cos(3x) + \frac{1}{9} \sin(3x)\right) + C \\ &= -\frac{1}{10} e^{-x} \left(3 \cos(3x) + \sin(3x)\right) + C\ . \end{aligned}$$

4. (15 points) Evaluate the integral

$$\int \frac{1}{(x-1)^2\,(x^2+1)}\,dx\ .$$

We expand the integrand into partial fractions:

$$\frac{1}{(x-1)^2\,(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}\ .$$

To find A, B, C we multiply through by $(x-1)^2(x^2+1)$ to get

$$\begin{aligned} 1 &= A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2 \\ &= A(x^3-x^2+x-1) + B(x^2+1) + (Cx+D)(x^2-2x+1) \\ &= x^3(A+C) + x^2(-A+B-2C+D) + x(A+C-2D) + (-A+B+D) \end{aligned}$$

Comparing coefficients, we get the following system of linear equations

$$\begin{array}{rrrrcl} A & & +C & & = & 0 \\ -A & +B & -2C & +D & = & 0 \\ A & & +C & -2D & = & 0 \\ -A & +B & & +D & = & 1 \end{array}$$

Subtracting the third from the first equation, we get $D=0$.
Subtracting the second from the fourth equation, we get $2C=1$, i.e. $C=1/2$.
Thus the first equation gives $A=-1/2$, and the fourth equation gives $B=1/2$.
We obtain

$$\begin{aligned} \int \frac{1}{(x-1)^2\,(x^2+1)}\,dx &= \int \left(-\frac{1}{2}\,\frac{1}{x-1} + \frac{1}{2}\,\frac{1}{(x-1)^2} + \frac{x/2}{x^2+1}\right)\,dx \\ &= \left(-\frac{1}{2}\,\ln|x-1| - \frac{1}{2}\,\frac{1}{x-1} + \frac{1}{4}\,\ln(x^2+1)\right) + C\ . \end{aligned}$$

5. (15 points) Evaluate the integral

$$\int \frac{x^2}{\sqrt{16-x^2}}\,dx\ .$$

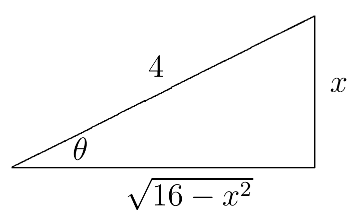
We use (inverse) trigonometric substitution:

$$x = 4\,\sin\theta, \quad dx = 4\,\cos\theta\;d\theta\ .$$

We obtain

$$\begin{aligned}
\int \frac{x^2}{\sqrt{16-x^2}}\,dx &= \int \frac{16\,\sin^2\theta}{4\,\cos\theta}\,4\,\cos\theta\;d\theta \\
&= \int 16\,\sin^2\theta\;d\theta \\
&= \int 8(1-\cos 2\theta)\;d\theta \\
&= 8\theta - 4\,\sin 2\theta + C \\
&= 8\theta - 8\,\sin\theta\,\cos\theta + C
\end{aligned}$$

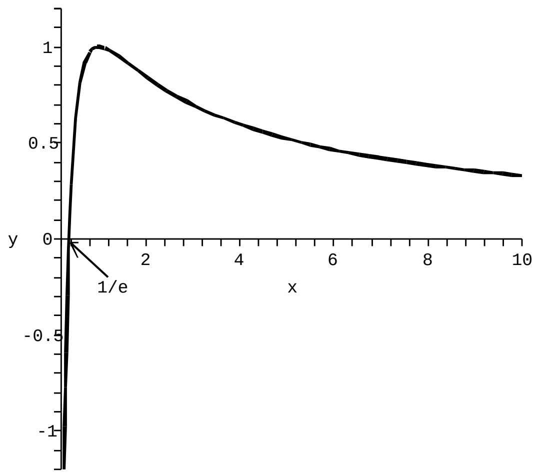
Now we use a triangle to get back to the original variable x.



So

$$\begin{aligned}
\int \frac{x^2}{\sqrt{16-x^2}}\,dx &= 8\,\sin^{-1}\left(\frac{x}{4}\right) - 8\,\frac{x}{4}\,\frac{\sqrt{16-x^2}}{4} + C \\
&= 8\,\sin^{-1}\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + C\ .
\end{aligned}$$

6. (10 points) Find the area between $y = \dfrac{1+\ln x}{x}$ and the x-axis for $\dfrac{1}{e} \leq x < \infty$.



We need to evaluate the improper integral

$$\int_{1/e}^{\infty} \frac{1+\ln x}{x}\, dx \; .$$

We use the substitution $u = 1 + \ln x$, $du = 1/x\ dx$.
This leads to the new integration limits:
$x = 1/e \Longrightarrow u = 0$, and $x \to \infty \Longrightarrow u \to \infty$.
We obtain

$$\int_{1/e}^{\infty} \frac{1+\ln x}{x}\, dx \;\; = \;\; \int_0^{\infty} u\, du$$

and

$$\begin{aligned} \int_0^{\infty} u\, du &= \lim_{t\to\infty} \int_0^t u\, du \\ &= \lim_{t\to\infty} u^2/2 \,\Big|_0^t \\ &= \lim_{t\to\infty} t^2/2 \;=\; \infty \; . \end{aligned}$$

Hence the improper integral diverges and the area is infinite.