Solutions to homework 1

1.5#1 Misère version of the take-away game. There are 21 chips, we can remove 1, 2, or 3.

Last player to move loses, hence position 1 is a P-position, from positions 2,3, and 4 we can move to 1, hence these are N-positions. Now, 5 is a P-position again, since we can only move to N-positions from it, and so on... We can guess the pattern by the first couple sample points: if $x \equiv 1 \mod 4$ then it is P-position, else an N-position. We can prove this by showing that (i) the terminal position is an N-position (misère); (ii) If we make a move from a P-position ($x \equiv 1 \mod 4$), then $x - 1 \not\equiv 1 \mod 4$, $x - 2 \not\equiv 1 \mod 4$, nor $x - 3 \not\equiv 1 \mod 4$; (iii) Similarly, if $y \not\equiv 1 \mod 4$, then one of $y - 1$, $y - 2$, $y - 3$ will be congruent to 1 mod 4.

Since, $21 \equiv 1 \mod 4$, the second player can always move to P-position during the game, and force the first player to lose.

1.5#4 Subtraction games

(a) $S = \{1, 3, 5, 7\}$. In this case $P = \{0, 2, 4, \ldots, 2k, \ldots\} = \{2k|k \in \mathbb{N}\}$, the even numbers. To prove this we only have to note that using a step from S we always jump from even to odd numbers.

(b) $S = \{1, 3, 6\}$. After computing a couple of positions, we guess that $P = \{x|x \equiv 0, 2, \text{or } 4 \mod 9\}$. We need to check that $p + s \notin P$, $\forall p \in P$,$\forall s \in S$, and also that $\exists s \in S$ such that $\forall n \notin P$: $n + s \in P$.

(c) $S = \{1, 2, 4, 8, 16, \ldots\}$. $P = \{3k|k \in \mathbb{N}\}$. Note that powers of 2 are not divisible by 3 and that subtracting 1 or 2 of any number not divisible by 3 will result in a number divisible by 3.

(d) $100 \equiv 0 \mod 2$ hence a P-position, 2nd player wins (in fact cannot lose). $100 \equiv 1 \mod 9$ hence an N-position, 1st player wins. $100 \equiv 1 \mod 3$ hence an N-position, 1st player wins.

2.6#2 Find all winning moves of nim

(a)
$$\begin{array}{rcr} 12 & = & 1100_2 \\ 19 & = & 10011_2 \\ 27 & = & 11011_2 \\ \hline \text{nim-sum} & = & 00100_2 \end{array}$$

Hence, we have to remove 1 from the third column and the only way we can do it is by subtracting 4 chips from the pile of 12, leaving 8.

(b)
$$\begin{array}{rcr} 13 & = & 1101_2 \\ 17 & = & 10001_2 \\ 19 & = & 10011_2 \\ 23 & = & 10111_2 \\ \hline \text{nim-sum} & = & 11000_2 \end{array}$$

The winning moves here are to remove 8 chips from one of the piles with 17, 19, or 23 chips.

(c) Since in both cases (a) and (b) we have at least two piles of size greater than one, the winning moves are the same even if the misère version of the game is played.

2.6#3 Nimble

The following bijection shows that this game is equivalent to nim. Each coin in nimble corresponds to pile in nim in the following sense. The position of the coin, equals the size of the pile. A move in nimble (moving a coin to left, i.e., a smaller position) equals to a move in nim, reducing the size of a pile. The terminal position in nimble, all coins are on the square labelled 0, corresponds to all empty piles in nim.

Hence, the position in nimble with the 6 coins corresponds to the following nim setup:

$$\begin{array}{rcr} 4 & = & 100_2 \\ 8 & = & 1000_2 \\ 8 & = & 1000_2 \\ 9 & = & 1001_2 \\ 10 & = & 1010_2 \\ 13 & = & 1101_2 \\ \hline \text{nim-sum} & = & 1010_2 \end{array}$$

The nim-sum is not zero, hence this is an N-position, next player wins. There are 5 winning moves. One is, for example, removing all chips from the pile of size 10.

2.6#7 Moore's Nim_k

(a) Nim_k with $k = 2$.

$$\begin{array}{rcr} 4 & = & 100_2 \\ 8 & = & 1000_2 \\ 8 & = & 1000_2 \\ 9 & = & 1001_2 \\ 10 & = & 1010_2 \\ 13 & = & 1101_2 \\ \hline \text{Nim}_3\text{-sum} & = & 2212_3 \end{array}$$

Since, the sum is not zero, by Moore's theorem this is an N-position. A winning move is removing 1 and 6 chips from the piles of size 8, leaving 7 and 2, respectively.

(b) To prove Moore's theorem, we need to show the characteristic property defined recursively by the three statements (i), (ii), and (iii).

(i) There is only one terminal position: $(0, \ldots, 0)$ and it is a P-position.

(ii) If we make a move from a P-position we cannot end up in a P-position again. In a P-position the Nim_k sum is zero. Given an arbitrary move, consider the leftmost column where one of the summands have changed. Note that in that column all changes must be 1 to 0 changes (in order for the move to be legal). Since the sum was congruent to 0 mod $k+1$ and we can only change at most k ones to zeros, after the move the sum in this column might be $1, 2, \ldots, k$ mod $k+1$, but not 0 mod $k+1$, hence we moved to an N-position.

(iii) From each N-position we can move to a P-position. We select k rows (corresponding to the piles) that we want to change in the following way. Consider the leftmost column in which the Nim_k sum is $s \not\equiv 0 \bmod k+1$. Select q_1 rows that have ones in this column and proceed to the right till the next column with non-zero Nim_k sum (ignoring the rows we have selected), and select additional q_2 rows. Continue this process until the number of added rows would exceed k, in which case pick only as many rows, so that the total number of picked rows is exactly k.

By our choice of rows we ensured the following property. For each column two things can happen. The unselected rows either have no ones in them. This happens when in each of these rows the most significant digit occurs to the right. In this case we set all digits in the selected rows to zero. The other option is that the Nim_k sum of unselected rows is $r \not\equiv 0 \bmod k+1$. In this case we set the sum in the selected rows to be $k+1-r$. Note, $1 \leq k+1-r \leq k$. We can do this last step, since the digits are already set to ones or can be set to ones by construction (recall that in the latter case the most significant digits in the selected rows are either in this column or to the left).

(c) The optimal play in the misère version of Nim_k is the following. Until there are at least $k+1$ piles of size larger than one play as under the normal rules. If there are k piles or less of size more than one, reduce all these piles to zero or one such that the number of piles of size one will be congruent to 1 mod $k+1$. This will eventually reduce the game to a take-away game with $S = \{1, 2, \ldots, k\}$, where the P-positions are exactly these: $P = \{x | x \equiv 1 \bmod k+1\}$.

With the above modifications, the proof for the $k = 1$ case can be generalized to our case.