Solutions to homework 2

4.5#2 One-pile game with the following rules:

You may remove (1) any number of chips divisible by three provided it is not the whole pile, or (2) the whole pile, but only if it has 2 (mod 3) chips.

The terminal positions are zero, one, and three. They have Sprague-Grundy value 0. From 2 we can only move to 0, by move (2), hence $g(2) = 1$. From 4 we can only move to 1, by move (1), hence $g(4) = 1$. From 5 we can move to 2 and 0, by moves (1) and (2), respectively, hence $g(5) = 2$, and so on. The first 15 SG values are the following:

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$g(x)$	0	0	1	0	1	2	1	2	3	2	3	4	3	4	5

We can observe that there is a pattern. Disregarding the first three positions (0, 1, 2) three consecutive numbers are increasing by 1 and the next three are also increasing in the same fashion, but the first element is 1 more than that of the previous triple. Formally, $g(0) = 0$ and $\forall n > 0$ we have that $g(n) = \lfloor \frac{n}{3} \rfloor - 1 + (n \bmod 3)$. Here, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

4.5#4 Kayles problem of Dudeney and Loyd

Of 13 pins in a row the second has been knocked down.

(a) In order to show that this is an N-position, we have to prove that the SG value at this position is $g(x) \neq 0$. Note that we have a sum of two games, one with 1 pin and another with 11 pins. Using Table 4.1 to look up the SG values, we obtain that $g(x) = g(1) \oplus g(11) = 1 \oplus 6 = 7 \neq 0$.

(b) A solution is to knock down pin 6. Then $g(x') = g(1) \oplus g(3) \oplus g(7) = 1 \oplus 3 \oplus 2 = 0$.

4.5#8 Rims

Rims is a disguised form of nim with the additional possibility of dividing the pile we removed chips from into two parts. Consider a position in Rims. We can group the points to classes, if they can be reached from each other with a continuous line without crossing any loops. Then we can think of each group of points as a nim pile. The rules are the same as in nim, except that in Rims we allow an additional move: to split a pile we reduced into two smaller piles. This happens when by drawing a closed loop we end up with some points inside and some points outside the loop.

The strategy to play Rims is the same as for nim. We just have to show that they agree on the P and N-positions. The P-positions in Rims are the ones where the nim-sum of the numbers of points in each group is zero.

Obviously, if there are no points left, we are in a terminal, and hence a P-position.

From every N-position we can move to a P-position exactly as we would play nim. That is, connect as many dots as many chips we would remove from a pile and make sure not to split the remaining points into different sets. All remaining points in that pile should be either all inside the loop or all outside.

Finally, we need to check that from any P-position we can only move to N-position. If the move we made was a nim move, this is clear. If not, say, we split the kth pile. This means that we changed the nim sum from $x_1 \oplus \cdots \oplus x_k \oplus \cdots \oplus x_m$ to $x_1 \oplus \cdots \oplus (a \oplus b) \oplus \cdots \oplus x_m$. Note that since there is no carry in nim-sum, for any $x_k = a + b$, we have that $a + b \geq a \oplus b$. As far as the nim-sum is concerned, this Rims splitting move corresponds to a legal move in nim (replacing x_k by a smaller value $a \oplus b$). Since P-positions in nim are the ones with nim-sum zero, we get that the same holds for Rims.