

Solutions to homework 3

5.5#1 Coin turning games

We analyze the following position under different coin turning game rules:

position	1	2	3	4	5	6	7	8	9
	T	T	H	T	H	H	T	T	H

Using the decomposition principle we can represent the game as a (disjunctive) sum of 4 games each exactly one head:

$$\begin{aligned}
 g(\text{TTHTHHTTH}) &= g(\text{TTHTTTTTT}) \oplus g(\text{TTTTHTTTT}) \oplus g(\text{TTTTTHTTT}) \oplus g(\text{TTTTTTTTH}) \\
 &= g(3) \oplus g(5) \oplus g(6) \oplus g(9).
 \end{aligned}$$

Note, in part (a) we use this numbering, i.e., starting from 0, e.g., $g(\text{TTH}) = g(3)$, but in parts (b)-(d) we shift the indexes by one, e.g., $g(\text{TTH}) = g(2)$.

- (a) A move in Turning Turtles is turning over a coin from heads to tails and, in addition, if desired, turning over one other coin to the left of it. Note, this is essentially nim, the height of each pile is represented by the location of the corresponding coin. Note, that $g(x) = x$. Hence, the SG-value: $g(\text{TTHTHHTTH}) = g(3) \oplus g(5) \oplus g(6) \oplus g(9) = 3 \oplus 5 \oplus 6 \oplus 9 = 9 \neq 0$. So, it is an N-position, the only winning move is to flip the coin on position 9.
- (b) A move in Twins is to flip two coins, the rightmost being a heads to tails flip. This is also nim, but with the indexes shifted by one, i.e., $g(x) = x$ (with the new numbering). Therefore, $g(\text{TTHTHHTTH}) = g(2) \oplus g(4) \oplus g(5) \oplus g(8) = 2 \oplus 4 \oplus 5 \oplus 8 = 11$. It is an N-position, the winning move is to flip the coins on positions 8 (from heads to tails) and 3 (from tails to heads).
- (c) In a Subtraction game with subtraction set $S = \{1, 3, 4\}$, when exactly two coins must be turned over a legal move is the following. Turn two coins over, one in position x (heads to tails) and the other in position $x - 1$, $x - 3$ or $x - 4$. Note, if we always require each move to turn two coins (even if $x \leq 4$, compare with example in page 29) then the SG-values for the first 9 positions are the following:

x	0	1	2	3	4	5	6	7	8
$g(x)$	0	1	0	1	2	3	2	0	1

 Hence, $g(2) \oplus g(4) \oplus g(5) \oplus g(8) = 0 \oplus 2 \oplus 3 \oplus 1 = 0$. This is a P-position, no winning moves here.
- (d) In Mock Turtles a legal move is to flip one, two or three coins, the rightmost being a heads to tails flip. Looking up the SG-values from page 30 we have that: $g(2) \oplus g(4) \oplus g(5) \oplus g(8) = 4 \oplus 8 \oplus 11 \oplus 16 = 23$. This is an N-position, a winning move is to flip coin 8 from heads to tails and coin 3 from tails to heads (since $g(3) = 7$ and $4 \oplus 8 \oplus 11 \oplus 7 = 0$).

5.5#6 Nim arithmetic

- (a) By using the Fermat 2-powers (2, 4, 16, 256, ...) and the distributive property we get that

$$\begin{aligned}
 6 \otimes 21 &= (4 \oplus 2) \otimes (16 \oplus 4 \oplus 1) = (4 \otimes 16) \oplus (4 \otimes 4) \oplus (4 \otimes 1) \oplus (2 \otimes 16) \oplus (2 \otimes 4) \oplus (2 \otimes 1) = \\
 &= 64 \oplus 6 \oplus 4 \oplus 32 \oplus 8 \oplus 2 = 64 \oplus 32 \oplus 8 = 104.
 \end{aligned}$$

- (b) Using similar techniques

$$\begin{aligned}
 25 \otimes 40 &= (16 \oplus 8 \oplus 1) \otimes (32 \oplus 8) = (16 \oplus (2 \otimes 4) \oplus 1) \otimes (16 \oplus 4) \otimes 2 = \\
 &= ((16 \otimes 16) \oplus (16 \otimes 4) \oplus (8 \otimes 16) \oplus ((2 \otimes 4) \otimes 4) \oplus (1 \otimes 16) \oplus (1 \otimes 4)) \otimes 2 = \\
 &= (24 \oplus 64 \oplus 128 \oplus (2 \otimes 6) \oplus 16 \oplus 4) \otimes 2 = ((24 \oplus 16) \oplus 64 \oplus 128 \oplus (2 \otimes (2 \oplus 4)) \oplus 4) \otimes 2 = \\
 &= (8 \oplus 64 \oplus 128 \oplus 3 \oplus 8 \oplus 4) \otimes 2 = (64 \oplus 128 \oplus 3 \oplus 4) \otimes 2 = \\
 &= (64 \otimes (1 \oplus 2) \oplus 3 \oplus 4) \otimes 2 = ((64 \otimes 3) \oplus 3 \oplus 4) \otimes 2 = 64 \oplus 1 \oplus 8 = 73
 \end{aligned}$$

- (c) For $15 \circlearrowleft 14$ we use Table 5.2 on page 33 to find the position in column 14 where the value is 15, i.e., we are solving the $x \otimes 14 = 15$ equation. And since $12 \otimes 15 = 14$, we have that $15 \circlearrowleft 14 = 12$.
- (d) Again we use table lookup. This time we are looking for y such that $y \otimes y = 8$. Clearly, we have to scan the diagonal for the number 8 and get that the square root of 8 is $y = 14$.
- (e) To solve $x^2 \oplus x \oplus 6 = 0$, we use the fact that the positive integers form a field under the nim addition and nim multiplication. And we know that there are two solutions to this equations, say, a and b , therefore we can factor the LHS as

$$x^2 \oplus x \oplus 6 = (x \oplus a) \otimes (x \oplus b) = x^2 \oplus (a \oplus b) \otimes x + a \otimes b.$$

We obtain the second equality by applying the distributive law. Now, we have that $a \oplus b = 1$ and $a \otimes b = 6$. The first condition means that a and b differ only in their last digit, in other words, they are consecutive numbers (the even number being the smaller).

Hence, we need to find two consecutive numbers (the even being smaller) whose nim product is 6. Alternatively, we could have arrived at this conclusion, by observing that the equation we are trying to solve is equivalent to $(x \oplus 1) \otimes x = 6$. Now, we need to scan the table searching for an entry with value 6 in the positions “one above” the diagonal, that is with coordinates $(2k, 2k + 1)$. We find that $14 \otimes 15 = 6$, hence $x = 14$, $x = 15$ are the solutions for the equation.

5.5#7 Tartan Theorem

Consider a position in Turning Corners with a single heads at (v_1, v_2) . It has SG-value $v_1 \otimes v_2$. Consider also a position in $G_1 \times G_2$ with a single heads at (x, y) , where the SG-values for position x in G_1 and position G_2 are $g_1(x) = v_1$ and $g_2(y) = v_2$, respectively.

Assume the Tartan Theorem (for $G_1 \times G_2$) holds for all $(x', y') \neq (x, y)$ with $x' \leq x, y' \leq y$.

- (a) First, we assume that there exists a move from the position in Turning Corners to a position of SG-value u . In other words, there is a move: flipping coins at (a, b) , (v_1, b) and (a, v_2) , such that $0 \leq a < v_1, 0 \leq b < v_2$ and $g(a, b) \oplus g(v_1, b) \oplus g(a, v_2) = u$.

Since $0 \leq a < v_1$, there must be a legal move in G_1 , say, turning coins a_1, \dots, a_m over that takes x (with SG-value $g_1(x) = v_1$) to a position with SG-value a . Similarly, we have that flipping coins b_1, \dots, b_n takes position y to another position in G_2 with SG-value b .

As a consequence, there is a legal move in $G_1 \times G_2$, namely flipping coins (a_i, b_j) , (x, b_j) and (a_i, y) , for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Note that by construction:

$$\begin{aligned} g(a_1, b_1) \oplus \dots \oplus g(a_m, b_n) &= (g_1(a_1) \otimes g_2(b_1)) \oplus \dots \oplus (g_1(a_m) \otimes g_2(b_n)) = \\ &= (g_1(a_1) \oplus \dots \oplus g_1(a_m)) \otimes (g_2(b_1) \oplus \dots \oplus g_2(b_n)) = a \otimes b = g(a, b) \end{aligned}$$

Here, the first and fourth equality is true by the induction hypothesis, the second follows from the distributive property of the nim product and the nim addition, and the third follows from the fact that the SG-value of a position with multiple heads is just the nim sum of the SG-values.

Similarly, for $g(x, b_1) \oplus \dots \oplus g(x, b_n) = g(x, b)$ and $g(a_1, y) \oplus \dots \oplus g(a_m, y) = g(a, y)$. Hence, this move in $G_1 \times G_2$ has SG-value $g(a, b) \oplus g(x, b) \oplus g(a, y) = u$, as required.

The proof of the other direction is similar. Assume that there exists a move from the position (x, y) with $g_1(x) = v_1$ and $g_2(y) = v_2$ in $G_1 \times G_2$ to a position of SG-value u , namely, turning coins over at $\{a_1, \dots, a_m, x\} \times \{b_1, \dots, b_n, y\}$, takes (x, y) position to a position with SG-value u . Then, let $a = g_1(a_1) \oplus \dots \oplus g_1(a_m)$ and $b = g_2(b_1) \oplus \dots \oplus g_2(b_n)$. Flipping coins (a, b) , (a, y) , (x, b) , (x, y) is a move in Turning Corners and the equations above certify that $g(a, b) \oplus g(x, b) \oplus g(a, y) = u$. Therefore, we have a corresponding move in Turning Corners with value u . It is not necessary a *legal* move, since it is possible that $g_1(a) = a > g_1(x)$, i.e., (x, y) might not be the SE corner.

- (b) We have shown in part (a) that we can move to a position of SG-value u in $G_1 \times G_2$ if and only if we can move in Turning Corners to a position with SG-value u . In Turning Corners the SG-value for the position with a single heads at (v_1, v_2) is $v_1 \otimes v_2$. By the property of the mex function, for any $u : 0 \leq u < v_1 \otimes v_2$, we can reach some position with SG-value u using legal moves.

Since, we allowed all possible (not just legal) moves in Turning Corners we also have to show that we cannot have SG-value $u = v_1 \otimes v_2$ value after any move (even if $a > v_1$ and/or $b > v_2$). Since,

$$(a \otimes b) \oplus (a \otimes v_2) \oplus (v_1 \otimes b) = v_1 \otimes v_2$$

is equivalent to

$$(a \otimes b) \oplus (a \otimes v_2) \oplus (v_1 \otimes b) \oplus (v_1 \otimes v_2) = 0$$

$$(a \oplus v_1) \otimes (b \oplus v_2) = 0,$$

which can only happen if $a \oplus v_1 = 0$ or $b \oplus v_2 = 0$. This is not possible, since $a \neq v_1$ and $b \neq v_2$.

5.5#8 (Mock Turtles) \times Ruler

- (a) Using the SG-values for Mock Turtles:

$$\begin{array}{rcccccc} x: & 0 & 1 & 2 & 3 & 4 \\ g_1(x): & 1 & 2 & 4 & 7 & 8 \end{array}$$

and the SG-values for Ruler:

$$\begin{array}{rcccccccc} y: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ g_2(y): & 1 & 2 & 1 & 4 & 1 & 2 & 1 & 8 \end{array}$$

and using nim multiplication we can create the SG-table for the (Mock Turtles) \times Ruler :

	0	1	2	3	4	5	6	7
0	1	2	1	4	1	2	1	8
1	2	3	2	8	2	3	2	12
2	4	8	4	6	4	8	4	11
3	7	9	7	10	7	9	7	15
4	8	12	8	11	8	12	8	13

- (b) The position

T	H	T	T	T	T	T	T
T	T	T	T	T	T	T	T
T	T	T	T	T	T	T	T
T	T	T	T	T	T	T	T
T	T	T	T	T	T	T	H

has SG value $(1 \otimes 2) \oplus (8 \otimes 8) = 2 \oplus 13 = 15$.

A winning move is to flip 9 coins at $\{1, 2, 5\} \times \{6, 7, 8\}$, i.e., the coins in the intersection of the last three columns with the first two and last rows. This move results the following position:

T	H	T	T	T	H	H	H
T	T	T	T	T	H	H	H
T	T	T	T	T	T	T	T
T	T	T	T	T	T	T	T
T	T	T	T	T	H	H	T

which is indeed a P-position since it has SG-value: $2 \oplus 2 \oplus 1 \oplus 8 \oplus 3 \oplus 2 \oplus 12 \oplus 12 \oplus 8 = 0$.

6.4#1 Green Hackenbush

The SG-values of the graphs in Figure 6.8 (in order from left to right) are 2, 3, 2 and 5. Their nim sum is $3 \oplus 5 = 6$, hence we are in an N-position. A winning move is, for example, to cut off a path of length two from the right branch of the last image, changing its SG-value to 3.