Solutions to homework 4

1.5#2 Player I holds a black Ace and a red 8. Player II holds a red 2 and a black 7. If the chosen cards are of the same color Player I wins, if they differ Player II does. The amount won is the sum of the cards.

The payoff function is given by the following matrix:

$$A = \begin{pmatrix} -3 & 8 \\ 10 & -15 \end{pmatrix}$$

If we denote the probability of Player I putting the Ace by p, then the *equalizing* strategy tells us that

$$-3p + 10(1-p) = 8p - 15(1-p).$$

$$10 - 13p = 23p - 15.$$

From which $p = \frac{25}{36}$. Hence, Player I's optimal strategy is choosing the Ace with probability $\frac{25}{36}$ and choosing 8 with probability $\frac{11}{36}$.

Similarly, if we denote the probability of Player II chosing the red 2 by q, we get

$$-3q + 8(1-q) = 10q - 15(1-q).$$

$$8 - 11q = 25q - 15.$$

Hence, $q = \frac{23}{36}$. Therefore, Player II's optimal strategy is choosing the Ace with probability $\frac{23}{36}$ and putting 8 with probability $\frac{13}{36}$.

The value of the game is $v = \frac{25}{36} \cdot (-3) + \frac{11}{36} \cdot 10 = \frac{-75+110}{36} = \frac{35}{36}$.

1.5#3 Sherlock Holmes

The payoff function is given by the matrix:

$$A = \begin{pmatrix} 100 & -50 \\ 0 & 100 \end{pmatrix}.$$

Let the probability for Moriarty to stop at Canterbury (i.e., selecting row 1) be p. Then we can find the optimal strategy by solving

$$100p = -50p + 100(1-p),$$

which gives us $p = \frac{2}{5}$. And similarly, Sherlock Holmes' optimal strategy (given that the probability of him getting off at Canterbury is q, i.e., the probability of selecting column 1) is given by the solution of

$$100q - 50(1-q) = 100(1-q),$$

which is $q = \frac{3}{5}$.

The value of the game to Moriarty is $v = \frac{2}{5} \cdot 100 + \frac{3}{5} \cdot 0 = 40$.

2.6#2 Solving a 2×2 game with one parameter

$$A = \begin{pmatrix} 0 & 2 \\ t & 1 \end{pmatrix}.$$

There are three cases to consider. (i) Assume first that $t \leq 0$. Then $a_{11} = 0$ is a saddle point, since it is the minimum value in the first row, and the maximum value in the first column. The value of the game is 0. (ii) Now, assume that $0 < t \leq 1$. Now, $a_{21} = t$ is a saddle point, being the minimum of the second row and maximum of the first column. The value of the game is t. (iii) Finally, let $t > 1$.

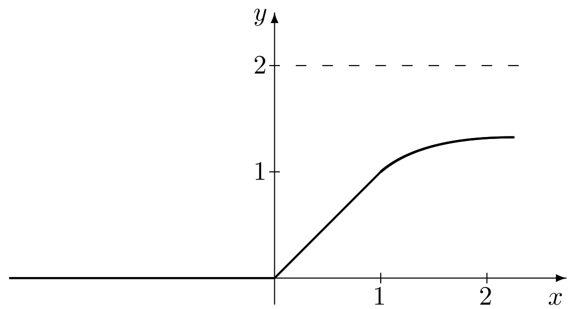

Figure 1: The graph of the function $v(t)$. The dashed line $y = 2$ represents the asymptote of the function.

There is no saddle point, since $0 < 2 > 1 < t > 0$ holds. By the formula for solving with equalizing strategies (see page II – 10) the value is:

$$\frac{ac - bd}{a - b + c - d} = \frac{0 - 2t}{0 - 2 + 1 - t} = \frac{2t}{t + 1}.$$

Figure 1 shows the graph of value of the game $v(t) = \begin{cases} 0 & \text{if} \quad t \leq 0 \\ t & \text{if} \quad 0 < t \leq 1 \\ \frac{2t}{t+1} & \text{if} \quad 1 < t \end{cases}$.

2.6#4 Reduce by dominance to 2×2 games and solve.

(a)

$$\begin{pmatrix} 5 & 4 & 1 & 0 \\ 4 & 3 & 2 & -1 \\ 0 & -1 & 4 & 3 \\ 1 & -2 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 5 & 4 & 1 & 0 \\ 0 & -1 & 4 & 3 \\ 1 & -2 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 1 & 0 \\ -1 & 4 & 3 \\ -2 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 1 & 0 \\ -1 & 4 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 0 \\ -1 & 3 \end{pmatrix}$$

In the first step we removed the second row which was dominated by the first. In the second step we removed the first column which was dominated by the second column. In the third step we removed the last row which was dominated by the second row. In the last step we removed the second column which was dominated by the last column.

The resulting 2×2 matrix has no saddle points. Hence, by the equalizing strategies (see pages II – 9, 10) we have that

$$p = \frac{3 - (-1)}{(4 - 0) + (3 - (-1))} = \frac{1}{2},$$

and

$$q = \frac{3 - 0}{(4 - 0) + (3 - (-1))} = \frac{3}{8}.$$

Therefore, the optimal strategies are $(1/2, 0, 1/2, 0)$ for Player I, and $(0, 3/8, 0, 5/8)$ for Player II. The value of the game is $v = \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot (-1) = \frac{3}{2}$.

(b)

$$\begin{pmatrix} 10 & 0 & 7 & 1 \\ 2 & 6 & 4 & 7 \\ 6 & 3 & 3 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 10 & 0 & 7 \\ 2 & 6 & 4 \\ 6 & 3 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 10 & 0 & 7 \\ 2 & 6 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 10 & 0 \\ 2 & 6 \end{pmatrix}$$

Here, in the first step we removed the last column which was dominated by the second column. Then we removed the last row which was dominated by the average of the first two. And finally, we removed the last column which was dominated by the average of the first two.

The resulting 2×2 matrix has no saddle points. So, by the same method as in part (a) we get that

$$p = \frac{6-2}{(10-0)+(6-2)} = \frac{2}{7},$$

and

$$q = \frac{6-0}{(10-0)+(6-2)} = \frac{3}{7}.$$

Hence, the optimal strategies for Player I and II are $(2/7, 5/7, 0)$ and $(3/7, 4/7, 0, 0)$, respectively. The value of the game is $v = \frac{3}{7} \cdot 10 + \frac{4}{7} \cdot 0 + 0 \cdot 7 + 0 \cdot 1 = \frac{30}{7}$.

2.6#7 Sure-fire test

Let

$$A = \begin{pmatrix} 5 & 8 & 3 & 1 & 6 \\ 4 & 2 & 6 & 3 & 5 \\ 2 & 4 & 6 & 4 & 1 \\ 1 & 3 & 2 & 5 & 3 \end{pmatrix}$$

and consider the mixed strategies $\mathbf{p} = (6/37, 20/37, 0, 11/37)^\mathsf{T}$ and $\mathbf{q} = (14/37, 4/27, 0, 19/37, 0)^\mathsf{T}$. Since, $\mathbf{p}^\mathsf{T} A = (3, 3, 3, 3, 3)$, we have that Player I has a strategy to win at least 3 (independent of Player II's strategy) and $A\mathbf{q} = (3, 3, 3, 3)^\mathsf{T}$ implies that Player II has a strategy not to lose more than 3. Since, the two values are equal, both strategies are optimal and the value of the game is $v = 3$.

2.6#10 Magic Square Games

Similarly, to the Latin Square Games (pages II – 13, 14) the value of an $n \times n$ Magic Square is the average of the numbers in a row, and the optimal strategy for both player is to chose each pure strategy with probability $1/n$. We show that these strategies satisfy the conditions of optimality.

The value is the average of all n^2 numbers in the matrix, i.e., $\frac{1}{n} \sum_{k=1}^{n} k = \frac{n+1}{2}$. In particular, since in a Magic Square all row sums and column sums are equal, the value is also equal to the average of the numbers in a row or a column. Hence, the above described mixed strategy is optimal for both players.

For example, let

$$A = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}.$$

We have $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})A = (\frac{17}{2}, \frac{17}{2}, \frac{17}{2}, \frac{17}{2})$ and $A(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^\mathsf{T} = (\frac{17}{2}, \frac{17}{2}, \frac{17}{2}, \frac{17}{2})^\mathsf{T}$. Therefore, the two strategies are indeed optimal. The value of the game is $V = \frac{17}{2}$.