# Solutions to homework 5

**3.7#1** Consider the game with matrix

$$A = \begin{pmatrix} -2 & 2 & -1 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}.$$

(a) $a_{23} = 1$ is a saddle point, since it is simultaneously the minimum value of the second row and the maximum of the third column.

(b) Since $\det A = -2 + 6 + 0 - 2 - 0 - (-3) = 5 \neq 0$, $A$ is invertible. The inverse $A$ can be computed in different ways. For example, by performing Gauss elimination on the matrix and mimicking the same operations on the identity matrix:

$$\left(\begin{array}{rrr|rrr} -2 & 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_1,R_2]{\text{swap}} \left(\begin{array}{rrr|rrr} 1 & 1 & 1 & 0 & 1 & 0 \\ -2 & 2 & -1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3'=3R_1-R_3]{R_2'=R_2+2R_1}$$

$$\to \left(\begin{array}{rrr|rrr} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 4 & 1 & 1 & 2 & 0 \\ 0 & 3 & 2 & 0 & 3 & -1 \end{array}\right) \xrightarrow{R_2'=R_2/4} \left(\begin{array}{rrr|rrr} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1/4 & 1/4 & 1/2 & 0 \\ 0 & 3 & 2 & 0 & 3 & -1 \end{array}\right) \xrightarrow[R_3'=R_3-3R_2]{R_1'=R_1-R_2}$$

$$\to \left(\begin{array}{rrr|rrr} 1 & 0 & 3/4 & 1/4 & 1/2 & 0 \\ 0 & 1 & 1/4 & 1/4 & 1/2 & 0 \\ 0 & 0 & 5/4 & -3/4 & 3/2 & -1 \end{array}\right) \xrightarrow[R_2'=R_2-\frac{1}{5}R_3, R_3'=\frac{4}{5}R_3]{R_1'=R_1-\frac{3}{5}R_3} \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 1/5 & -2/5 & 3/5 \\ 0 & 1 & 0 & 2/5 & 1/5 & 1/5 \\ 0 & 0 & 1 & -3/5 & 6/5 & -4/5 \end{array}\right).$$

Hence, $A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 1 \\ -3 & 6 & -4 \end{pmatrix}$.

(c) Consider, $\mathbf{q} = (\frac{1}{6}, \frac{1}{2}, \frac{1}{3})^{\mathsf{T}}$. This is an optimal strategy (see (2) on page II – 17 for the definition), since, no matter which row Player I choses the payoff is no more than the value $V = 1$:

$$\mathbf{Aq} = \begin{pmatrix} \frac{1}{3} \\ 1 \\ \frac{5}{6} \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Caution, not all strategies of Player II with positive values will work! For example, $\mathbf{q} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^{\mathsf{T}}$ does not. If Player II were to play this strategy then Player I could increase his winning to 4/3 by playing the last strategy with probability 1.

(d) The equation (16) does only give an optimal strategy for Player II if the assumption that Player "I has an optimal strategy giving positive weight to each of the rows." holds (first paragraph of section 3, page II –27). Since this assumption is false in our case we cannot use equation (16) to compute the optimal strategy of Player II. It can be seen by a short computation that Player I has only 1 optimal strategy, giving positive weight to only one of the rows. Since the only solution to $\mathbf{p}^{\mathsf{T}}\mathbf{A} \geq (1,1,1)$ is $\mathbf{p}^{\mathsf{T}} = (0,1,0)$.

**3.7#2** Diagonal games

Let $\mathbf{A}$ be a matrix of a diagonal game:

$$\begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_m \end{pmatrix}.$$

(a) If $d_i = 0$ for some $i \in \{1, \ldots, n\}$ then $\mathbf{A}$ has a saddle point, namely this point and hence the value of the game is zero.

(b) If $d_i > 0$ and $d_j < 0$, then $a_{ij} = 0$ is a saddle point, since

$$\min\{a_{i1}, \ldots, a_{in}\} = \min\{0, \ldots, 0, d_i, 0, \ldots, 0\} = 0 = \max\{0, \ldots, 0, d_j, 0, \ldots, 0\} = \max\{a_{1j}, \ldots, a_{nj}\},$$

and thus the value of the game is zero.

(c) If $d_i < 0$, for all $i \in \{1, \ldots, n\}$ then the matrix is nonsingular and $\mathbf{1}^\mathsf{T}\mathbf{A}^{-1}\mathbf{1} \neq 0$, hence by Theorem 3.2 on page II – 20 the value of the game is $V = 1/\mathbf{1}^\mathsf{T}\mathbf{A}^{-1}\mathbf{1} = \frac{1}{\sum_{i=1}^{n} 1/d_i} < 0$.

**3.7#3** Player II chooses a number $j \in \{1, 2, 3, 4\}$, and Player I tries to guess what number II has chosen. If he guesses correctly and the number was $j$, he wins $2^j$ dollars from II. Otherwise there is no payoff. The matrix of the game is:

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 16 \end{pmatrix}.$$

Since $\mathbf{A}$ is nonsingular and $\mathbf{1}^\mathsf{T}\mathbf{A}^{-1}\mathbf{1} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} \neq 0$.

By Theorem 3.2 the value of the game is $V = \frac{16}{15}$ and the optimal strategies are

$$\mathbf{p}^\mathsf{T} = V\mathbf{1}^\mathsf{T}\mathbf{A}^{-1} = \left(\frac{8}{15}, \frac{4}{15}, \frac{2}{15}, \frac{1}{15}\right), \quad \mathbf{q} = V\mathbf{A}^{-1}\mathbf{1} = \begin{pmatrix} 8/15 \\ 4/15 \\ 2/15 \\ 1/15 \end{pmatrix}.$$

**3.7#8** Solving games given their matrices

(a)

$$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

The solution can be computed from the set of equations

$$\begin{array}{rcrcrcl} p_1 & & & & & = & V \\ -p_1 & + & 2p_2 & & & = & V \\ -p_1 & + & p_2 & + & 3p_3 & = & V \end{array}$$

$p_1 = p_2 = V$ and $p_3 = V/3$. We also have that $p_1 + p_2 + p_3 = 7V/3 = 1 \Rightarrow V = \frac{3}{7}$. Hence, $p_1 = p_2 = \frac{3}{7}$ and $p_3 = \frac{1}{7}$.

The optimal strategy for Player II can be computed in a similar fashion:

$$\begin{array}{rcrcrcl} q_1 & - & q_2 & - & q_3 & = & \frac{3}{7} \\ & & 2q_2 & + & q_3 & = & \frac{3}{7} \\ & & & & 3q_3 & = & \frac{3}{7} \end{array}$$

$q_1 = \frac{5}{7}$ and $q_2 = q_3 = \frac{1}{7}$.

(b)

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & \frac{3}{2} & 1 & 1 \\ 1 & 1 & \frac{4}{3} & 1 \\ 1 & 1 & 1 & \frac{5}{4} \end{pmatrix}, \quad \mathbf{A}' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

Note that the game with matrix $\mathbf{A}$ has the same optimal strategies for Players I and II as the game with matrix $\mathbf{A}' = \mathbf{A} - \mathbf{1}$. The only difference is that the value decreases by 1. A similar situation is explained on page II – 20. There it was used to create a nonsingular matrix, but here $\mathbf{A}$ is already nonsingular to begin with, but with this trick we can reduce the problem to solving the game on a diagonal matrix which is much simpler.

Since, $\mathbf{A}'$ is a diagonal matrix with $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ on its diagonal, we have that the value of the game is $V' = 1/(1+2+3+4) = \frac{1}{10}$. This implies that the value of the original game is $\frac{11}{10}$. The optimal strategy for both players is $\left(\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}\right)$.

(c) Note this matrix is singular. Fortunately, we can reduce it by to a $3 \times 3$ matrix by dominance. First observe that the last column is dominated by the first, and then the last row is dominated by the average of the first two rows.

$$\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 1 & 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Now, we need to solve the game of the resulting $3 \times 3$ matrix. Since it is diagonal, the value of this game $V = \frac{1}{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}} = \frac{12}{13}$. This is also the value of the original game. The optimal strategy for both Player I and II is $\left(\frac{6}{13}, \frac{4}{13}, \frac{3}{13}\right)$, and hence, $\left(\frac{6}{13}, \frac{4}{13}, \frac{3}{13}, 0\right)$ for the original game.

**3.7#9** Another Mendelsohn game

(a) Let $n \geq 5$. The game has the following $n \times n$ payoff matrix

$$\mathbf{A} = \begin{pmatrix} 0 & -2 & 1 & 1 & \dots & 1 \\ 2 & 0 & -2 & 1 & \dots & 1 \\ -1 & 2 & 0 & -2 & \dots & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ -1 & \dots & -1 & 2 & 0 & -2 \\ -1 & \dots & -1 & -1 & 2 & 0 \end{pmatrix}. \text{ Formally, } a_{ij} = \begin{cases} 1, & \text{if } i < j-1 \\ -2, & \text{if } i = j-1 \\ 0, & \text{if } i = j \\ 2, & \text{if } i = j+1 \\ -1, & \text{if } i > j+1 \end{cases}$$

(b) As pointed out in the exercise the optimal strategy satisfies $p_i > 0$ for $i = 1, \dots, 5$ and $p_i = 0$ for all other $i$. Therefore, it is sufficient to consider the following $5 \times 5$ matrix:

$$\tilde{\mathbf{A}} = \begin{pmatrix} 0 & -2 & 1 & 1 & 1 \\ 2 & 0 & -2 & 1 & 1 \\ -1 & 2 & 0 & -2 & 1 \\ -1 & -1 & 2 & 0 & -2 \\ -1 & -1 & -1 & 2 & 0 \end{pmatrix}.$$

This matrix is skew-symmetric ($\tilde{\mathbf{A}} = -\tilde{\mathbf{A}}^{\mathsf{T}}$), so the game is symmetric. Hence, by Theorem 3.3 it has value zero.

To compute the optimal strategies we have to solve:

$$\begin{array}{rcrcrcrcrcl} & - & 2p_2 & + & p_3 & + & p_4 & + & p_5 & = & 0 \\ 2p_1 & & & - & 2p_3 & + & p_4 & + & p_5 & = & 0 \\ -p_1 & + & 2p_2 & - & & - & 2p_4 & + & p_5 & = & 0 \\ -p_1 & - & p_2 & + & 2p_3 & + & & - & 2p_5 & = & 0 \\ -p_1 & - & p_2 & - & p_3 & + & 2p_4 & & & = & 0 \end{array}$$

Note if we subtract equation 3 from the sum of equations one and five we get that: $-5p_2+5p_4 = 0$, implying that $p_2 = p_4$. From this and equation 3 we also get that $p_1 = p_5$. Substituting this information back to equations one and two (the other equations do not give any extra information) and using the fact that $p_1 + p_2 + p_3 + p_4 + p_5 = 1$ we can reduce the problem to the following 3 equations:

$$\begin{array}{rcrcrcl} p_1 & - & p_2 & + & p_3 & = & 0 \\ 3p_1 & + & p_2 & - & 2p_3 & = & 0 \\ 2p_1 & + & 2p_2 & + & p_3 & = & 1 \end{array}$$

Since,

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 3 & 1 & -2 \\ 2 & 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{16} \begin{pmatrix} 5 & 3 & 1 \\ -7 & -1 & 5 \\ 4 & -4 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{16} \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix},$$

we have that the optimal solution for Player I is $\tilde{\mathbf{p}} = \left(\frac{1}{16}, \frac{5}{16}, \frac{1}{4}, \frac{5}{16}, \frac{1}{16}\right)^{\mathsf{T}}$ for the $5 \times 5$ game and hence $\mathbf{p}^{\mathsf{T}} = \left(\frac{1}{16}, \frac{5}{16}, \frac{1}{4}, \frac{5}{16}, \frac{1}{16}, 0, \ldots, 0\right)$ for the original $n \times n$ case. This works for Player II as well, since the game is symmetric (see Theorem 3.3).

Finally, we check that this is an optimal solution indeed. Since $\mathbf{p}$ has only five components which are nonzero, the payoff is the same for $n \times n$ matrices as for the top left $5 \times 5$ matrix, formally:

$$\mathbf{p}^{\mathsf{T}} \mathbf{A} \mathbf{p} = \tilde{\mathbf{p}}^{\mathsf{T}} \tilde{\mathbf{A}} \tilde{\mathbf{p}} = 0.$$

**3.7#10** Silverman Games

We assume $n \geq 5$ as in the previous game but, in fact, $n \geq 3$ would be sufficient.

The general form of the matrix is the following:

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 2 & 2 & 2 & 2 & 2 & \ldots \\ 1 & 0 & -1 & -1 & -1 & 2 & 2 & \ldots \\ -2 & 1 & 0 & -1 & -1 & -1 & -1 & \ldots \\ -2 & 1 & 1 & 0 & -1 & -1 & -1 & \ldots \\ -2 & 1 & 1 & 1 & 0 & -1 & -1 & \ldots \\ -2 & -2 & 1 & 1 & 1 & 0 & -1 & \ldots \\ -2 & -2 & 1 & 1 & 1 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \text{ Formally, } a_{ij} = \begin{cases} 2, & \text{if} \quad j \in [3i, \infty) \\ -1, & \text{if} \quad j \in [i+1, 3i) \\ 0, & \text{if} \quad j = i \\ 1, & \text{if} \quad j \in (i/3, i-1] \\ -1, & \text{if} \quad j \in (-\infty, i/3] \end{cases}$$

(a) The game is symmetric (since $a_{ij} = -a_{ji}$ for all $i, j$). It is easy to see that row one dominates all rows $5, 6, \ldots$, and so on, and row five dominates rows three and four. Since $\mathbf{A}$ is skew-symmetric a similar argument applies to the columns. This way we reduced the game to the following $3 \times 3$ matrix:

$$\begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix}.$$

(b) Note that this is the reduced Mendelsohn Game (see page II – 24) with solution $\mathbf{p}^{\mathsf{T}} = \mathbf{q}^{\mathsf{T}} = (1/4, 1/2, 1/4)$, hence the solution of the Silverman games (for $n \geq 5$) is just $(1/4, /1/2, 0, 0, 1/4, 0, \ldots)$ and value $V = 0$.