Solutions to homework 6

2.5#1 Strategic Equilibria Are Individually Rational Consider a two-person noncooperative general-sum game with corresponding $m \times n$ payoff matrices $\mathbf{A}$ and $\mathbf{B}$. Let $(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$ be a strategic equilibrium. This implies that $\tilde{\mathbf{p}}^\top \mathbf{A}\tilde{\mathbf{q}} \geq \mathbf{p}^\top \mathbf{A}\tilde{\mathbf{q}}$ for all $\mathbf{p} \in \mathbf{X}^*$. Furthermore, it implies that

$$\tilde{\mathbf{p}}^\top \mathbf{A}\tilde{\mathbf{q}} \geq \max_{\mathbf{p}\in\mathbf{X}^*} \mathbf{p}^\top \mathbf{A}\tilde{\mathbf{q}}.$$

Trivially, for every $\mathbf{p} \in \mathbf{X}^*$ we have that

$$\mathbf{p}^\top \mathbf{A}\tilde{\mathbf{q}} \geq \min_{\mathbf{q}\in\mathbf{Y}^*} \mathbf{p}^\top \mathbf{A}\mathbf{q}.$$

Hence,

$$\tilde{\mathbf{p}}^\top \mathbf{A}\tilde{\mathbf{q}} \geq \max_{\mathbf{p}\in\mathbf{X}^*} \min_{\mathbf{q}\in\mathbf{Y}^*} \mathbf{p}^\top \mathbf{A}\mathbf{q} = \max_{\mathbf{p}\in\mathbf{X}^*} \min_{1\leq j\leq n} \sum_{i=1}^{m} p_i a_{ij} = v_I.$$

The first equality follows from equation (7) on page II – 36, the second is just the definition of v_I.

The proof for the fact that $\tilde{\mathbf{p}}^\top \mathbf{B}\tilde{\mathbf{q}} \geq v_{II}$ is almost identical.

2.5#2 Find the safety levels, the MM-strategies, and all SE's and associated vector payoffs of the following games in strategic form

(a) The bimatrix $\begin{pmatrix} (0,0) & (2,4) \\ (2,4) & (3,3) \end{pmatrix}$ can be represented by $\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 4 \\ 4 & 3 \end{pmatrix}$.

Since $\mathbf{A}$ has a saddle point at $a_{21} = 2$ the safety level of Player I is $v_I = \text{Val}(\mathbf{A}) = 2$ and the MM-strategy is $\mathbf{p} = (0,1)^\top$.

$\mathbf{B}^\top$ does not have a saddle point, so the safety level of Player II is

$$v_{II} = \text{Val}(\mathbf{B}^\top) = \frac{0 \cdot 3 - 4 \cdot 4}{0 - 4 + 3 - 4} = \frac{16}{5},$$

and its MM-strategy is $\mathbf{q} = (1/5, 4/5)$.

Using the labeling method described in page III – 11 we get $\begin{pmatrix} (0,0) & (2,4^*) \\ (2^*,4^*) & (3^*,3) \end{pmatrix}$ and hence there is one PSE at $(2,1)$ with payoff $(2,4)$. This is the only SE in fact, since if we try to solve for the equalizing strategy we get $q_1 = \frac{3-2}{0-2+3-2} = -1 < 0$.

(b) The bimatrix $\begin{pmatrix} (1,4) & (4,1) \\ (2,2) & (3,3) \end{pmatrix}$ can be represented by $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.

Again, $\mathbf{A}$ has a saddle point $a_{21} = 2$. The safety level of Player I is $v_I = \text{Val}(\mathbf{A}) = 2$ and its MM-strategy is $\mathbf{p} = (0,1)^\top$.

$\mathbf{B}^\top$ does not have a saddle point, so the safety level of Player II is

$$v_{II} = \text{Val}(\mathbf{B}^\top) = \frac{4 \cdot 3 - 2 \cdot 1}{4 - 1 + 3 - 2} = \frac{5}{2},$$

and its MM-strategy is $\mathbf{q} = (1/2, 1/2)^\top$.

Using the labeling method we get $\begin{pmatrix} (1,4^*) & (4^*,1) \\ (2^*,2) & (3,3^*) \end{pmatrix}$ and hence there is no PSE. However, there is an equalizing SE given by $\mathbf{p} = (1/4, 3/4)^\top$ and $\mathbf{q} = (1/2, 1/2)^\top$ with payoff $(\mathbf{p}^\top \mathbf{A}\mathbf{q}, \mathbf{p}^\top \mathbf{B}\mathbf{q}) = (5/2, 5/2)$.

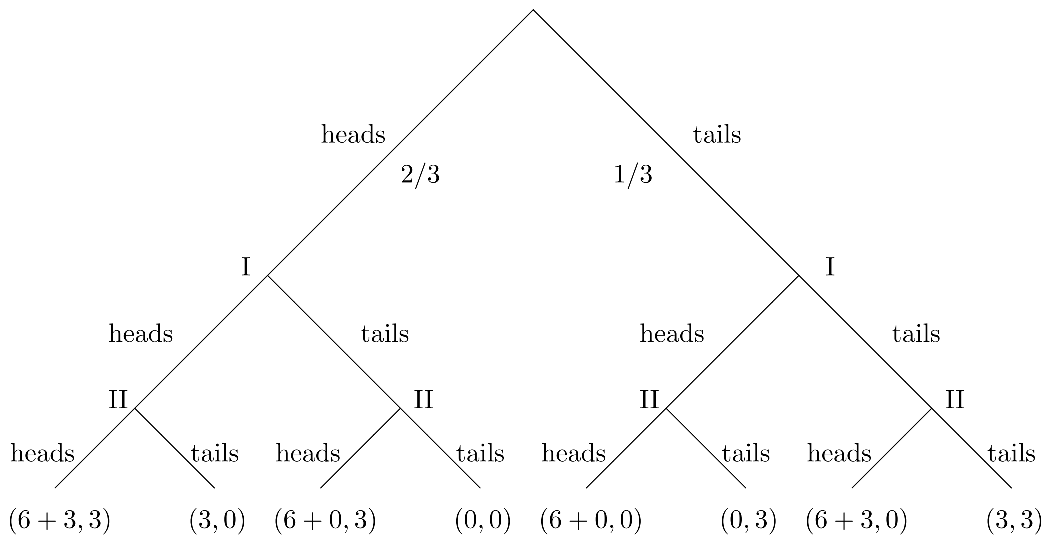

Figure 1: The Kuhn tree of the game.

(c) The bimatrix $\begin{pmatrix} (0,0) & (0,-1) \\ (1,0) & (-1,3) \end{pmatrix}$ can be represented by $\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & -1 \\ 0 & 3 \end{pmatrix}$. Since $\mathbf{A}$ has a saddle point at $a_{12} = 0$ the safety level of Player I is $v_I = \text{Val}(\mathbf{A}) = 0$ and its MM-strategy is $\mathbf{p} = (1,0)^\top$. Also $\mathbf{B}^\top$ has a saddle point at $b_{11} = 0$ (note $\mathbf{B}$ does not have a saddle point there!). So, the safety level of Player II is $v_{II} = \text{Val}(\mathbf{B}^\top) = 0$ and its MM-strategy is $\mathbf{q} = (1,0)^\top$.

Using the labeling method we get $\begin{pmatrix} (0,0^*) & (0^*,-1) \\ (1^*,0) & (-1,3^*) \end{pmatrix}$ and hence there is no PSE. There is an equalizing SE given by $\mathbf{p} = (3/4, 1/4)^\top$ and $\mathbf{q} = (1/2, 1/2)^\top$ with payoff $(\mathbf{p}^\top \mathbf{A}\mathbf{q}, \mathbf{p}^\top \mathbf{B}\mathbf{q}) = (0,0)$.

2.5#4 An extensive form non-zero-sum game

(a) Figure 1 shows the Kuhn tree of the game.

(b) Player I has four pure strategies:

- claim heads when it the coin turned up heads, and claim tails when it turned up tails,
- claim tails when it is heads, and heads when it is tails,
- claim heads if it is heads and if it is tails,
- claim tails if it is heads and if it is tails.

Perhaps it is more natural to name these strategies as follows, X ={truth, lie, heads, tails}. Similarly, Player II has the following pure strategies Y ={trust, distrust, heads, tails}, based on how II reacts on I's claims.

Keeping this order of strategies, we can compute the strategic form of the game:

$$\begin{pmatrix} (7,3) & (5,0) & (9,2) & (3,1) \\ (2,0) & (4,3) & (6,2) & (0,1) \\ (8,2) & (2,1) & (8,2) & (2,1) \\ (1,1) & (7,2) & (7,2) & (1,1) \end{pmatrix}.$$

For example, value at position $(1,2)$ is computed in the following way. The strategies are: Player I tells the truth, and Player II distrusts Player I (always guesses the opposite). 2/3 of the time when the coin turns up heads, Player I claims heads, and Player II guesses tails in which case the payoff is \$3 for Player I for telling the truth, and \$0 for Player II not guessing the coin correctly. 1/3 of the time the coin turns up tails, Player I claims tails, Player II guesses heads in which case the payoff for Player I is \$9 for telling the truth and because Player II guessed heads, and again \$0 for Player II for not guessing the coin correctly. Hence, the payoff for playing these pure strategies is $\frac{2}{3}(3,0) + \frac{1}{3}(9,0) = (5,0)$.

(c) Using the labeling method we get

$$\begin{pmatrix} (7,3^*) & (5,0) & (9^*,2) & (3^*,1) \\ (2,0) & (4,3^*) & (6,2) & (0,1) \\ (8^*,2^*) & (2,1) & (8,2^*) & (2,1) \\ (1,1) & (7^*,2^*) & (7,2^*) & (1,1) \end{pmatrix},$$

hence there are two PSE's in this game.

2.5#7 Strategic Equilibria Survive Elimination of Strictly Dominated Strategies

Let $(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$ denote a Strategic Equilibrium.

By the assumption of strict domination, for all j there exist x_i with $0 \le x_i \le 1$ and $\sum_{i=2}^m x_i = 1$ such that $a_{1j} < \sum_{i=2}^m x_i a_{ij}$.

If each a_{1j} is dominated then so is their convex combination. Without loss of generality, we can weight the a_{1j}'s by $\tilde{\mathbf{q}}$, the optimal strategy of Player II, and get that

$$\sum_{j=1}^n a_{1j}\tilde{q}_j < \sum_{j=1}^n \left(\sum_{i=2}^m x_i a_{ij} \right) \tilde{q}_j = \sum_{j=1}^n \sum_{i=2}^m x_i a_{ij} \tilde{q}_j \le \sum_{j=1}^n \sum_{i=1}^m \tilde{p}_i a_{ij} \tilde{q}_j. \tag{1}$$

The first inequality is a consequence of the domination, and the second is just the optimality of a SE.

For $i \neq 1$ we use another inequality:

$$\sum_{j=1}^n a_{ij}\tilde{q}_j \le \sum_{j=1}^n \sum_{k=1}^m \tilde{p}_k a_{kj} \tilde{q}_j. \tag{2}$$

This is a simple consequence of the fact that if both Players I and II play their optimal strategy (RHS) they are not worse off than if Player I plays a pure strategy i and Player II plays optimally (LHS).

Now putting all together,

$$\begin{aligned} \sum_{i=1}^m \sum_{j=1}^n \tilde{p}_i a_{ij} \tilde{q}_j &= \sum_{i=1}^m \tilde{p}_i \left(\sum_{j=1}^n a_{ij}\tilde{q}_j \right) \\ &= \tilde{p}_1 \left(\sum_{j=1}^n a_{ij}\tilde{q}_j \right) + \sum_{i=2}^m \tilde{p}_i \left(\sum_{j=1}^n a_{ij}\tilde{q}_j \right) \\ &\le \tilde{p}_1 \left(\sum_{j=1}^n \sum_{i=1}^m \tilde{p}_i a_{ij}\tilde{q}_j \right) + \sum_{i=2}^m \tilde{p}_i \left(\sum_{j=1}^n \sum_{k=1}^m \tilde{p}_k a_{kj}\tilde{q}_j \right) \\ &= \left(\tilde{p}_1 + \sum_{i=2}^m \tilde{p}_i \right) \left(\sum_{j=1}^n \sum_{i=1}^m \tilde{p}_i a_{ij}\tilde{q}_j \right) \\ &= \sum_{i=1}^m \sum_{j=1}^n \tilde{p}_i a_{ij}\tilde{q}_j \end{aligned}$$

Hence, it must be equality throughout, but since (1) is strict this is only possible if $\tilde{p}_1 = 0$.