- Notations:

(1) Integer partition: $\lambda = (\lambda_1, \dots, \lambda_\ell)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$

(2) $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$

(3) $\ell(\lambda) :=$ # positive parts in λ

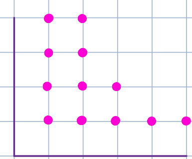
(4) Partial order: $\lambda \leq \mu$ iff $|\lambda| = |\mu|$ and $\lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k \ \forall k \in \mathbb{Z}^+$

(5) (French style) Ferrers shape of $\lambda := \{(i,j) \mid 1 \leq j \leq \ell(\lambda),\ 1 \leq i \leq \lambda_j\}$

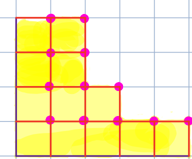
(6) lattice squares (boxes) of λ = boxes with NE corner in Ferrers shape of λ

(7) Shape generator of λ: $B_\lambda(q,t) := \sum_{(i,j)\in\lambda} q^{i-1} t^{j-1}$

Example: $\lambda = (5,3,2,2,0,0,0)$. We have
- $|\lambda| = 12$, $\ell(\lambda) = 4$
- Ferrers shape:



- Boxes:



- $\mu = (6,3,2,1,0,0)$.

  Then $\lambda \leq \mu$ b/c $5 \leq 6$, $5+3 \leq 6+3$, $5+3+2 \leq 6+3+2$, $5+3+2+1 = 6+3+2+1$

- $j=1$ (ie first row) $\quad j=2$ (ie second row)

$$B_\lambda(q,t) = t^0(q^0+q^1+q^2+q^3+q^4) + t^1(q^0+q^1+q^2) + t^2(q^0+q^1) + t^3(q^0+q^1)$$

$$= 1+q+q^2+q^3+q^4+t+qt+q^2t+t^2+qt^2+t^3+qt^3$$

(8) $k = \mathbb{Q}(q,t)$

(9) $\Lambda = \Lambda_k(X)$ $\quad$ ($X = x_1, x_2, \dots$ infinite variables): algebra of symmetric functions

(10) $\omega: \Lambda \to \Lambda$ s.t $s_\lambda \mapsto s_{\lambda^*}$ $\quad$ (λ^*: conjugate of λ)

(11) $\langle\ ,\ \rangle: \Lambda \times \Lambda \to k$ s.t $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$

(12) $n(\mu) = \sum_i (i-1)\mu_i$

(13) Modified Macdonald polynomial: $\widetilde{H}_\mu(X;q,t) = t^{n(\mu)} \overbrace{J_\mu\left[\frac{X}{1-t^{-1}}; q, t^{-1}\right]}^{J_\mu(X;q,t):\ \text{Macdonald polynomial}}$ $\qquad$ Sub $t = t^2$, $q = q^2$

(14) For $f \in \Lambda$, let $f[B]$ be the eigenoperator on the basis $\{\widetilde{H}_\mu\}$:

$$f[B]\widetilde{H}_\mu := f[B_\mu(q,t)]\widetilde{H}_\mu$$

Example:
- $n(5,3,2,2) = 0\times5 + 1\times3 + 2\times2 + 3\times2 = 13$
- $p_2[B]\widetilde{H}_{5322} = p_2[B_{5322}(q,t)]\widetilde{H}_{5322} = (1+q^2+q^4+q^6+q^8+t^2+q^2t^2+q^4t^2+t^4+q^2t^4+t^6+q^2t^6)\widetilde{H}_{5322}$.

(15) $\Delta_f = f[B]$ and $\Delta'_f = f[B-1]$

Example:
- $\Delta_{p_2}\widetilde{H}_{5322} = p_2[B]\widetilde{H}_{5322} = (1+q^2+q^4+q^6+q^8+t^2+q^2t^2+q^4t^2+t^4+q^2t^4+t^6+q^2t^6)\widetilde{H}_{5322}$.
- $\Delta'_{p_2}\widetilde{H}_{5322} = p_2[B-1]\widetilde{H}_{5322} = (q^2+q^4+q^6+q^8+t^2+q^2t^2+q^4t^2+t^4+q^2t^4+t^6+q^2t^6)\widetilde{H}_{5322}$.

Extended Delta Conjecture:

symmetric function side $\qquad$ Combinatorial side

$$\underbrace{\Delta_{h_\ell}\Delta'_{e_{k-1}} e_n}_{h_\ell[B]\, e_{k-1}[B-1]\, e_n} = \langle z^{n-k} \rangle \sum_{\lambda \in D_{n+\ell}} \sum_{P \in L_{n+\ell,\ell}(\lambda)} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(\lambda)} x^{\mathrm{wt}_+(P)} \prod_{r_i(\lambda) = r_{i-1}(\lambda)+1} (1 + z t^{-r_i(\lambda)})$$

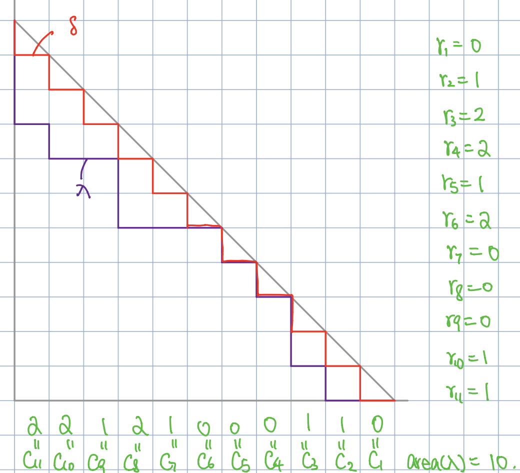
for $\ell \geq 0$ and $1 \leq k \leq n$.

Def: A Dyck path is a SE lattice path weakly below the line segment connecting $(0,N)$ and $(N,0)$.

Notation:
- D_N := set of all Dyck paths weakly below the line segment connecting $(0,N)$ and $(N,0)$
- δ is the highest Dyck path in D_N, i.e. $\underbrace{SESE\cdots SE}_{N \text{ pairs of } SE}$ from $(0,N)$ to $(N,0)$.
  staircase path
- $\text{area}(\lambda) := |\delta/\lambda|$ = # lattice squares above λ and below δ for $\lambda \in D_N$
- $r_i(\lambda)$ = # lattice squares in row i above λ and below δ (i: count from top to bottom) $(1 \le i \le N)$
  = distance from the i^{th} South step of λ to $x = i-1$ (i^{th} S-step of δ)
- $c_i(\lambda)$ = # lattice squares in column i above λ and below δ (i: count from right to left)
  = distance from the $(N+1-i)^{\text{th}}$ East step to $y = i-1$
  from $(0,N)$ to $(N,0)$

* $r_1(\lambda) = c_1(\lambda) = 0$, $r_i(\lambda) \le r_{i-1}(\lambda)+1$, $c_i(\lambda) \le c_{i-1}(\lambda)+1$ $\forall i \ge 1$, $\text{area}(\lambda) = r_1(\lambda)+\cdots+r_N(\lambda) = c_1(\lambda)+c_2(\lambda)+\cdots+c_N(\lambda)$.

e.g. $N=11$.

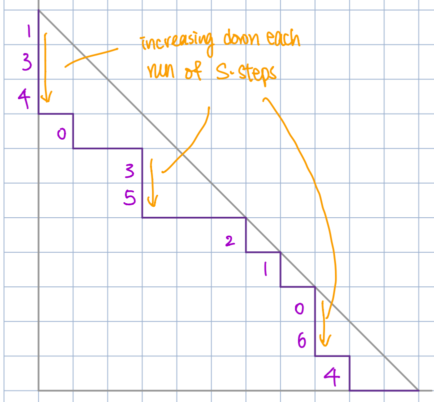


$\text{area}(\lambda) = 10$.

Def: A labelling $P = (P_1, \ldots, P_N) \in \mathbb{N}^N$ attaches a label in $\mathbb{N} = \{0, 1, \ldots\}$ to each South step of $\lambda \in D_N$ s.t. labels increase from N to S along each vertical runs.

A partial labelling is a labelling with no 0 on the last S-step.

e.g.



Notation:
- $L(\lambda) = L_N(\lambda)$ = set of all labellings of $\lambda \in D_N$
- $L_{N,\ell}(\lambda)$ = set of all partial labellings of $\lambda \in D_N$ with exactly ℓ '0's.
  * $L_{N,N}(\lambda) = \emptyset$ b/c the last S-step cannot be zero. Hence we may assume $0 \le \ell \le N-1$.

Given $P \in L(\lambda)$, define

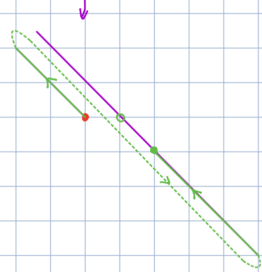
Fix a S-step, read SE down the next (lower) diag and count the number of smaller labels

$$\operatorname{dinv}(P) := \left|\{(i,j) : 1 \le i < j \le N,\ r_i(\lambda) = r_j(\lambda) \text{ and } P_i < P_j\}\right| + \left|\{(i,j) : 1 \le i < j \le N,\ r_i(\lambda) = r_j(\lambda)+1 \text{ and } P_i > P_j\}\right|$$

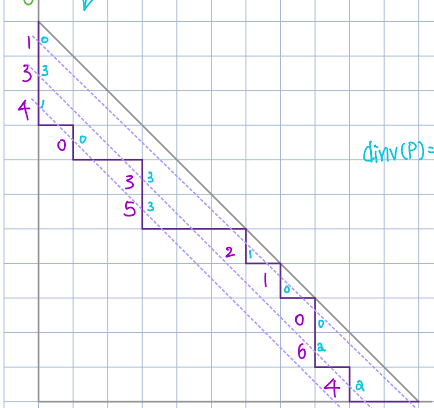
* Fix a S-step, read diag. "down" (i.e. SE direction ↘) and count the number of larger labels

Another way to count $\operatorname{dinv}(P)$:

Fix a S-step, run a "NW cycle" and count the number of smaller labels. Add up the numbers for all S-steps gives $\operatorname{dinv}(P)$



e.g.



$\operatorname{dinv}(P) = 15$

Given $P \in L(\lambda)$, define the weight of P

hence variables are $x_1, x_2, \dots$

$$x^{wt_+(P)} := \prod_{i \in [N],\ P_i \neq 0} x_{P_i} \qquad (\text{i.e. } wt_+(P) = \text{sequence whose } k^{th} \text{ coordinate is the } \# k \text{ in } P)$$

set $x_0 = 1$, $x^{wt(P)} = x^{wt_+(P)}$

hence variables are $x_0, x_1, \dots$

$$x^{wt(P)} := \prod_{i \in [N]} x_{P_i} \qquad (\text{i.e. } wt_+(P) = \text{sequence whose } k^{th} \text{ coordinate is the } \#(k-1) \text{ in } P)$$

Lemma 2.2.4: For any $\lambda \in D_N$, we have

$$\prod_{\substack{1 \le i \le N \\ \underbrace{r_i(\lambda) = r_{i-1}(\lambda)+1}}} (1 + z t^{-r_i(\lambda)}) = \prod_{\substack{1 \le i \le N \\ \underbrace{c_i(\lambda) = c_{i-1}(\lambda)+1}}} (1 + z t^{-c_i(\lambda)}).$$

i.e. λ has consecutive S-steps in rows $i-1$ and i (from top to bottom)

i.e. λ has consecutive E-steps in col $i-1$ and col i (from right to left)

Proof: Consider the word in $\{S, E\}$ listing steps in λ from $(0,N)$ to $(N,0)$.
Treat S's as the left parentheses, E's as the right parentheses
Pair S, E as parentheses (from outermost to innermost).
Since $\#S = \#E = N$, we can always pair them into N pairs of parentheses.

* For any double S, the left S must pair with the right of some double E steps.

((......)()
cannot happen as they would pair up first ⇒ must be ((......))
must have this

$\therefore$ We pair all $(\underbrace{i-1, i}_{\text{double } S})$ with $(\underbrace{j, j-1}_{\text{double } E})$, i.e. We pair up $i-1$ and $j-1$ with $\begin{cases} r_{i-1}(\lambda) = r_i(\lambda) - 1 \\ c_{j-1}(\lambda) = c_j(\lambda) - 1 \end{cases}$

$$\prod_{\substack{1 \le i \le N \\ r_i(\lambda) = r_{i-1}(\lambda)+1}} (1 - z t^{\underbrace{-r_{i-1}(\lambda)-1}_{-r_i(\lambda)}}) = \prod_{\substack{1 \le j \le N \\ c_j(\lambda) = c_{j-1}(\lambda)+1}} (1 - z t^{\underbrace{-c_{j-1}(\lambda)-1}_{-r_j(\lambda)}}) \Leftrightarrow \prod_{\substack{1 \le i \le N \\ r_i(\lambda) = r_{i-1}(\lambda)+1}} (1 - z t^{-r_i(\lambda)}) = \prod_{\substack{1 \le i \le N \\ c_i(\lambda) = c_{i-1}(\lambda)+1}} (1 - z t^{-c_i(\lambda)})$$

□

Extended Delta Conjecture:

symmetric function side

Combinatorial side

$$\underbrace{\Delta_{h_\ell}\Delta'_{e_{k-1}} e_n}_{h_\ell[B]\, e_{k-1}[B-1]\, e_n} = \langle z^{n-k} \rangle \sum_{\lambda \in D_{n+\ell}} \sum_{P \in L_{n+\ell,\ell}(\lambda)} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(\lambda)} x^{\mathrm{wt}_t(P)} \prod_{r_i(\lambda) = r_{i-1}(\lambda)+1} \left(1 + z t^{-r_i(\lambda)}\right) \quad \text{for } \ell \geq 0 \text{ and } 1 \leq k \leq n.$$

Set $N = n+\ell$, $m = k+\ell$, we have

$$h_\ell[B]\, e_{m-\ell-1}[B-1]\, e_{N-\ell}(X) = \langle z^{N-m} \rangle \sum_{\substack{\lambda \in D_N \\ P \in L_{N,\ell}(\lambda)}} q^{\mathrm{dinv}(P)}\, t^{|\delta/\lambda|}\, x^{\mathrm{wt}_t(P)} \prod_{\substack{1 < i \leq N \\ c_i(\lambda) = c_{i-1}(\lambda)+1}} \left(1 + z t^{-c_i(\lambda)}\right).$$