For any algebra A containing a copy of $\Lambda = \Lambda(X)$, there is an adjoint action of Λ on A arising from the Hopf algebra structure of Λ.

- X, Y : formal alphabet to distinguish $\Lambda \otimes \Lambda \cong \Lambda(X)\Lambda(Y)$
- coproduct : $\Delta f = f[X+Y]$

  antipode : $S(f) = f[-X]$

The adjoint action of $f \in \Lambda$ on $\zeta \in A$ is given by:

$$(\mathrm{Ad} f)\zeta = \sum_i f_i[Z]\,\zeta\, g_i[Z] \quad \text{where} \quad f[X-Y] = \sum_i f_i(X) g_i(Y) \leftarrow ((1\otimes S)\Delta f = \sum_i f_i \otimes g_i)$$

depends on what variable Λ in A contains

keep X

change Y to $-Y$

More explicitly, we have
- $(\mathrm{Ad}\, p_n)\zeta = p_n\zeta - \zeta p_n = [p_n, \zeta]$ $\quad (\because p_n[X-Y] = p_n(X) - p_n(Y) = p_n(X)\cdot 1 - 1\cdot p_n(Y))$
- $(\mathrm{Ad}\, h_n)\zeta = \sum_{j+k=n} (-1)^k h_j \zeta e_k$ $\quad (\because h_n[X-Y] = \sum_{j+k=n} h_j[X] h_k[-Y] = \sum_{j+k=n} h_j[X]\,(-1)^k e_k[Y])$

Let $\Omega(X) = \sum_n h_n(X)$. Then $(\mathrm{Ad}\,\Omega[zX])\zeta = \Omega[zX]\,\zeta\,\Omega[-zX]$. $(\because \Omega[z(X-Y)] = \sum_n h_n[zX - zY] = \sum_n \left(\sum_{j+k=n} h_j[zX] h_k[zY]\right) = \sum_j h_j[zX] \sum_k h_k[-zY] = \Omega[zX]\,\Omega[-zY])$

$= \exp \sum_k \frac{p_k(X)}{k}$

$\left(* \ \Omega[a_1 + a_2 + \cdots - b_1 - b_2 - \cdots] = \frac{\prod_i (1-b_i)}{\prod_i (1-a_i)}, \quad \Omega[XY] = \prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda s_\lambda[X] s_\lambda[Y] \quad (\text{Cauchy identity})\right)$

Notation: $M = (1-q)(1-t)$, $\hat{M} = (1 - \frac{1}{qt})(1-q)(1-t) = (1 - \frac{1}{qt})M$

- $k := \mathbb{Q}(q,t)$
- Schiffmann algebra $\mathcal{E}$: generated by
  - a central Laurent polynomial subalgebra $F = k[c_1^{\pm 1}, c_2^{\pm 1}]$
  - a family of subalgebras $\{\Lambda_F(X^{m,n}) : \text{g.c.d.}(m,n) = 1\}$ where $\Lambda_F(X^{m,n}) \cong \Lambda_F(X)$ for all coprime pairs (m,n)

Structure and symmetries :

- For $(m,n) \in \mathbb{Z}^2$ s.t. g.c.d.$(m,n) = 1$, a family of alphabets $\{X_\theta^{m,n} : \theta \in \arg(m+in)\}$ satisfying

$$X^{m,n}_{\theta+2\pi} = c_1^m c_2^n X^{m,n}_\theta \qquad (X^{m,n} := X^{m,n}_{\mathrm{Arg}(m+in)} \ \text{ i.e. } \ X^{m,n} = X^{m,n}_\theta \ \text{ with } \theta \in (-\pi, \pi])$$

e.g. $(m,n) = (3,-5)$, $\mathrm{Arg}(3-5i) = \tan^{-1}(-\frac{5}{3})$

$$X^{3,-5}_{\tan^{-1}(-\frac{5}{3})+4\pi} = c_1^3 c_2^{-5} X^{3,-5}_{\tan^{-1}(-\frac{5}{3})+2\pi} = c_1^6 c_2^{-10} X^{3,-5}$$

- $\Lambda_F(X^{m,n}) = \Lambda_F(X^{m,n}_\theta)$ (independent of the choice of θ)

  $\Lambda_K(X^{m,n}) \neq \Lambda_K(X^{m,n}_\theta)$ (depends on the choice of θ)

- Universal extension $\widehat{SL_2(\mathbb{Z})} \longrightarrow SL_2(\mathbb{Z})$ acts on the set

$$\{(m,n,\theta) \in (\mathbb{Z}^2 \setminus \{0\}) \times \mathbb{R} \mid \theta \text{ is a value of } \arg(m+in)\}$$

lifting the $SL_2(\mathbb{Z})$ action on (m,n), with the 'rotation by 2π' map $(m,n,\theta) \mapsto (m,n,\theta+2\pi)$ generating the central subgroup $\mathbb{Z}$.

$\widehat{SL_2(\mathbb{Z})}$ acts on $\mathcal{E}$ by K-algebra automorphisms, compatibly with $SL_2(\mathbb{Z})$-action on $\mathbb{Z}^2$.

$*$ $\widehat{SL_2(\mathbb{Z})}$ acts on the generators of $\mathcal{E}$ as follows:

$$\rho \cdot f(X^{m,n}_\theta) := f(X^{m',n'}_{\theta'}) \quad \text{for } f(X) \in \Lambda_K(X),\ \rho \in \widehat{SL_2(\mathbb{Z})}$$

where $\rho \cdot (m,n,\theta) = (m',n',\theta')$.

Action on F factors through the $SL_2(\mathbb{Z})$ action on $k \cdot \mathbb{Z}^2 \cong F$.

e.g. If $\rho \in \widehat{SL_2(\mathbb{Z})}$ is the 'rotation by 2π' element in $\widehat{SL_2(\mathbb{Z})}$, ρ fixes F, and

$\therefore \rho \sim$ multiply by $c_1^m c_2^n$ $\rightarrow$ $\rho \cdot f(X^{m,n}_\theta) = f(X^{m,n}_{\theta+2\pi}) = f[c_1^m c_2^n X^{m,n}_\theta]$.

same

$$\rho \cdot f(X^{m,n}_{\theta+2\pi}) = f(X^{m,n}_{\theta+4\pi}) = f[c_1^{2m} c_2^{2n} X^{m,n}_\theta], \quad \rho \cdot f(X^{m,n}_{\theta+2\pi}) = \rho \cdot f[c_1^m c_2^n X^{m,n}_\theta] = f[c_1^m c_2^n X^{m,n}_{\theta+2\pi}] = f[c_1^{2m} c_2^{2n} X^{m,n}_\theta]$$

- $\mathcal{E}$ is $\mathbb{Z}^2$ graded
- Central subalgebra F with degree $(0,0)$
- For $f \in \Lambda(X)$ with $\deg f = d$, $\deg(f(X^{m,n})) = (dm, dn)$. e.g. $\deg e_{21}(X^{3,-5}) = (9,-15)$

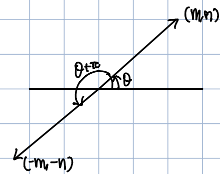
Note: 'rotation by 2π'-map ρ preserves all relations that respect the $\mathbb{Z}^2$-grading b/c ρ acts by multiplying $\underbrace{c_1^m c_2^n}_{\deg=(0,0)}$ in any given deg. (m,n).

Defining relations of $\mathcal{E}$:

(1) Heisenberg relations:

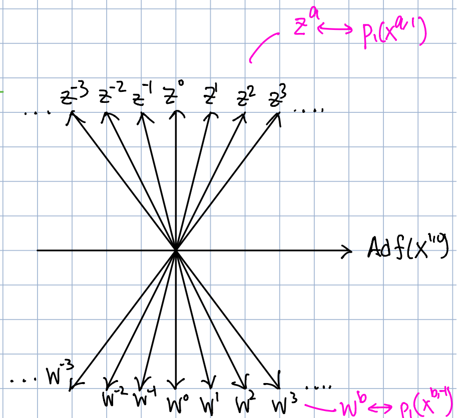
For each pair of subalgebras $\Lambda_F(X^{m,n})$ and $\Lambda_F(X^{-m,-n})$

$$[p_k(X_\theta^{-m,-n}),\ p_\ell(X_{\theta+\pi}^{m,n})] = \delta_{k\ell}\, k\, p_k\left[\frac{c_1^m c_2^n - 1}{\hat{M}}\right]$$



e.g. • $[p_2(X_\theta^{-3,-5}),\ p_4(X_{\theta+\pi}^{3,5})] = 0$ $\quad \theta \in \arg(-3-5i)$

• $[p_7(X_\theta^{-3,-5}),\ p_7(X_{\theta+\pi}^{3,5})] = 7p_7\left[\frac{c_1^3 c_2^5 - 1}{\hat{M}}\right] = 7\,\frac{c_1^{21}c_2^{35}-1}{(1-q^7)(1-t^7)(1-\frac{1}{q^7t^7})}$



(2) Internal action relations: (describes the adjoint action of each $\Lambda_F(X^{m,n})$ on $\mathcal{E}$)

(For simplicity, we set $(m,n)=(1,0)$. The full set of relations is closed under $\widehat{SL_2\mathbb{Z}}$ action)

$$(\mathrm{Ad}\, f(X^{1,0}))\,p_1(X^{m,1}) = (\omega f)[z]\Big|_{z^k \mapsto p_1(X^{m+k,1})}$$

$$(\mathrm{Ad}\, f(X^{1,0}))\,p_1(X^{m,-1}) = (\omega f)[-z]\Big|_{z^k \mapsto p_1(X^{m+k,-1})}$$

e.g. • $(\mathrm{Ad}\,p_2(X^{1,0}))p_1(X^{3,1}) = (\omega p_2)[z]\Big|_{z^k\mapsto p_1(X^{3+k,1})} = (-1)^{2-1}p_2[z]\Big|_{z^k\mapsto p_1(X^{3+k,1})} = -z^2\Big|_{z^k\mapsto p_1(X^{3+k,1})} = -p_1(X^{5,1})$

• $(\mathrm{Ad}\,e_3(X^{1,0}))p_1(X^{3,1}) = h_3[z]\Big|_{z^k\mapsto p_1(X^{3+k,1})} = \sum_{\lambda\vdash 3}\frac{p_\lambda[z]}{z_\lambda}\Big|_{z^k\mapsto p_1(X^{3+k,1})} = \frac{p_3[z]}{3} + \frac{p_{21}[z]}{2} + \frac{p_{111}[z]}{6}\Big|_{z^k\mapsto p_1(X^{3+k,1})}$

$= \frac{1}{3}z^3 + \frac{1}{2}z^2z + \frac{1}{6}z^3\Big|_{z^k\mapsto p_1(X^{3+k,1})} = p_1(X^{6,1})$

• $(\mathrm{Ad}\,h_3(X^{1,0}))\,p_1(X^{3,-1}) = e_3[z]\Big|_{z^k\mapsto p_1(X^{3+k,-1})} = \sum_{\lambda\vdash 3}\frac{(-1)^{3-\ell(\lambda)}}{z_\lambda}p_\lambda[z]\Big|_{z^k\mapsto p_1(X^{3+k,-1})}$

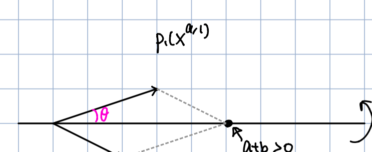
$= \frac{p_3[z]}{3} - \frac{p_{21}[z]}{2} + \frac{p_{111}[z]}{6}\Big|_{z^k\mapsto p_1(X^{3+k,-1})}$

$= \left(\frac{1}{3}-\frac{1}{2}+\frac{1}{6}\right)z^3\Big|_{z^k\mapsto p_1(X^{3+k,-1})} = 0$

(3) Axis-crossing relations:

(For simplicity, we set $(m,n)=(1,0)$. The full set of relations is closed under $\widehat{SL_2\mathbb{Z}}$ action)

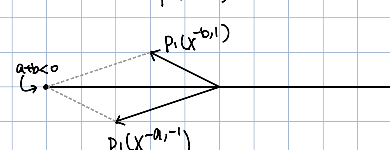
$$[p_1(X^{b,-1}),\ p_1(X^{a,1})] = -\frac{e_{a+b}[-\hat{M}X^{1,0}]}{\hat{M}} \quad \text{for } a+b>0$$



Rotate this relation by π anti-clockwise:

$$[p_1(X^{-b,1}),\ p_1(X_{\theta+\pi}^{-a,-1})] = -\frac{e_{a+b}[-\hat{M}X^{-1,0}]}{\hat{M}} \quad \text{for } a+b>0$$

$\theta = \tan^{-1}(\frac{1}{a})$



$\therefore\ [p_1(X^{-b,1}),\ p_1(X^{-a,-1})] = [p_1(X^{-b,1}),\ p_1(X_{\theta-\pi}^{-a,-1})]$

$= [p_1(X^{-b,1}),\ c_1^a c_2\, p_1(X_{\theta+\pi}^{-a,-1})] = -\frac{c_1^a c_2\, e_{a+b}[-\hat{M}X^{-1,0}]}{\hat{M}}$

i.e. $[p_1(X^{-a,-1}),\ p_1(X^{-b,1})] = \frac{c_1^a c_2\, e_{a+b}[-\hat{M}X^{-1,0}]}{\hat{M}}$ for $a+b>0$ $\underset{a\to -b,\ b\to -a}{\Rightarrow}$ $[p_1(X^{b,-1}),\ p_1(X^{a,1})] = \frac{c_1^{-b}c_2\, e_{-a-b}[-\hat{M}X^{-1,0}]}{\hat{M}}$ for $a+b<0$

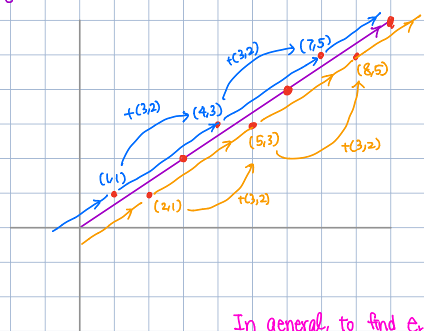
By Heinsenberg relation: $[p_1(x^{b,-1}), p_1(x^{-b,1})] = \dfrac{c_1^{-b}c_2 - 1}{\hat{M}}$

$\therefore$ We get

$$[p_1(x^{b,-1}), p_1(x^{a,1})] = \begin{cases} -\dfrac{e_{a+b}[-\hat{M}x^{1,0}]}{\hat{M}} & \text{for } a+b>0 \\ \dfrac{c_1^{-b}c_2 - 1}{\hat{M}} & \text{for } a+b=0 \\ \dfrac{c_1^{-b}c_2\, e_{-(a+b)}[-\hat{M}x^{-1,0}]}{\hat{M}} & \text{for } a+b<0 \end{cases}$$

* Using $\widehat{SL_2(\mathbb{Z})}$ image, we can express any $e_k[-\hat{M}x^{m,n}]$ with $n>0$ $(n<0)$ in terms of iterated commutators of the elements $p_1(x^{a,1})$. $(p_1(x^{a,-1}))$

e.g. Express $e_2[-\hat{M}x^{3,2}]$ in terms of $p_1(x^{a,1})$'s:



• points on $x=i$ $(i\in\mathbb{Z})$ closest to the line $y=\frac{3}{2}x$

These points are: $\{(1,1)+k(3,2) : k\in\mathbb{Z}\}$, $\{(2,1)+k(3,2) : k\in\mathbb{Z}\}$ or $\{(0,0)+k(3,2) : k\in\mathbb{Z}\}$.

$\therefore$ Points just above the axis $y=\frac{3}{2}x$ are: $\{(1,1)+k(3,2) : k\in\mathbb{Z}\}$

Points just below the axis $y=\frac{3}{2}x$ are: $\{(0,0)+k(3,2) : k\in\mathbb{Z}\}$

In general, to find $e_k[-\hat{M}x^{m,n}]$:
Find two closest points to the ray $y=\frac{n}{m}x$, say (a,b) and (c,d) s.t. $\gcd(a,b)=\gcd(c,d)=1$, $(a,b)+(c,d)=k(m,n)$

By axis-crossing relations:

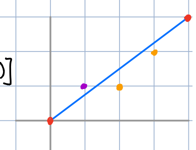
$(2,1)+(4,3)=(6,4)=2(3,2)$

(We can also choose $(1,1)+(5,3)$)

$$e_2[\hat{M}x^{3,2}] = -\hat{M}[\underbrace{p_1(x^{2,1})}_{\text{in } x^{a,1}}, p_1(x^{4,3})]$$

$$p_1(x^{4,3}) = -\frac{1}{\hat{M}}e_1[-\hat{M}x^{4,3}] = -\frac{1}{\hat{M}}\cdot(-\hat{M})[p_1(x^{3,2}), p_1(x^{1,1})]$$

$$= [p_1(x^{3,2}), p_1(x^{1,1})] \quad \text{in } x^{a,1}$$



$$p_1(x^{3,2}) = -\frac{1}{\hat{M}}e_1[-\hat{M}x^{3,2}] = -\frac{1}{\hat{M}}(-\hat{M})[p_1(x^{2,1}), p(x^{1,1})]$$

$$= [p_1(x^{2,1}), p_1(x^{1,1})]$$

Hence $e_2[-\hat{M}x^{3,2}] = -\hat{M}[p_1(x^{2,1}), [[p_1(x^{2,1}), p_1(x^{1,1})], p_1(x^{1,1})]]$.

Def: Define $\mathcal{E}^{*,>0}$:= subalgebra of $\mathcal{E}$ generated by $\Lambda_F(x^{m,n})$, $n>0$ (upper half subalgebra of $\mathcal{E}$)

$\mathcal{E}^{*,<0}$:= subalgebra of $\mathcal{E}$ generated by $\Lambda_F(x^{m,n})$, $n<0$ (lower half subalgebra of $\mathcal{E}$)

$\because$ $e_k[-\hat{M}x^{m,n}]$ can be written as iterated commutators of elements $p(x^{a,1})$, $a\in\mathbb{Z}$

$\therefore$ $\mathcal{E}^{*,>0}$ is generated by $\{p_1(x^{a,1}) : a\in\mathbb{Z}\}$.

Similarly, $\mathcal{E}^{*,<0}$ is generated by $\{p_1(x^{a,-1}) : a\in\mathbb{Z}\}$.

$\because$ The internal action relations give the adjoint action of $\Lambda_F(x^{1,0})$ on the space spanned by $\{p(x^{a,1}), p_1(x^{a,-1}) : a\in\mathbb{Z}\}$

$\therefore$ Using the formula $(\mathrm{Ad}\, f)(\zeta_1\zeta_2) = \sum((\mathrm{Ad}\, f_{(1)})\zeta_1)((\mathrm{Ad}\, f_{(2)})\zeta_2)$ where $\Delta f = \sum f_{(1)}\otimes f_{(2)}$, this determines the adjoint action on $\mathcal{E}^{*,>0}$ and $\mathcal{E}^{*,<0}$.

- The Heisenberg relations give the adjoint action of $\Lambda_F(X^{1,0})$ on $\Lambda(X^{-1,0})$, and $\Lambda(X^{1,0})$ acts on itself trivially

$(\mathrm{Ad} f(X^{1,0}))\, g(X^{1,0}) := (f[1]g)(X^{1,0})$

$\therefore$ All these determine the adjoint action of $\Lambda_F(X^{1,0})$ on $\mathcal{E}$. (Holds for $\Lambda(X^{m,n})$ by symmetry)

Useful relations: $[\omega p_k(X^{1,0}), p_1(X^{a,1})] = p_1(X^{a+k,1})$ ← By internal action relations (put $f = \omega p_k$)

$[\omega p_k(X^{-1,0}), p_1(X^{a,1})] = -p_1(X^{a-k,1})$ ← Use symmetry

Def: (Anti-involution) ← product reversing automorphism

$$\Phi : \mathcal{E} \to \mathcal{E} \quad \text{s.t.} \quad \Phi(g(c_1,c_2)) = g(c_2^{-1}, c_1^{-1}), \quad \Phi f(X_\theta^{m,n}) = f(X^{n,m}_{\frac{\pi}{2}-\theta})$$

Note: $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \longleftrightarrow \frac{\pi}{2} - \theta \in (0, \pi) \subseteq (-\pi, \pi)$

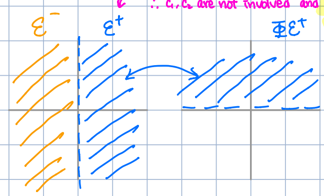
$\therefore$ c_1, c_2 are not involved and $\Phi f(X^{m,n}) = f(X^{n,m})$ $\forall f \in \Lambda_K(X^{m,n}) \subset \mathcal{E}^+$

Notation: (used in Path)

We leave out F b/c the relations of $\mathcal{E}^+$ do not depend on c_1, c_2. ← Heinsenburg, internal action and Axis-crossing relations

$\mathcal{E}^+ \subseteq \mathcal{E}$:= subalgebra of $\mathcal{E}$ generated by $\Lambda_K(X^{m,n})$, $m > 0$ ('right half-plane' subalgebra)

$\mathcal{E}^- \subseteq \mathcal{E}$:= subalgebra of $\mathcal{E}$ generated by $\Lambda_F(X^{m,n})$, $m < 0$ ('left half-plane' subalgebra)



Note: $\Phi\mathcal{E}^+$ is generated by $\Lambda_K(X^{m,n})$, $n > 0$