Notation: • $f^\bullet$ : operator of multiplication by $f$

$\therefore (\omega f)^\bullet \neq \omega \cdot f^\bullet$  (Note: $(\omega f)^\bullet = \omega \cdot f^\bullet \cdot \omega$)

e.g. $(\omega h_n)^\bullet e_m = e_n e_m$, $(\omega \cdot h_n^\bullet) e_m = \omega(h_n e_m) = e_n h_m$

• $f^\perp$ : $\langle\,,\,\rangle$-adjoint of $f^\bullet$  (Note: $(\omega f)^\perp = \omega \cdot f^\perp \cdot \omega$)

i.e. Suppose $f^\perp(g) = h$. That means

$$\langle u, h\rangle = \langle u, f^\perp(g)\rangle = \langle f^\bullet u, g\rangle \quad \forall u \in \Lambda$$

(e.g. $1^\perp(g) = g$ b/c $\langle 1\cdot u, g\rangle = \langle u, g\rangle \ \forall u \in \Lambda$)

(dual Pieri rule) $e_k^\perp(s_\lambda) = \sum_{\lambda/\mu \text{ is a vert. strip of size } k} s_\mu$ b/c $\langle e_k^\perp(s_\lambda), s_\mu\rangle = \langle s_\lambda, \underset{s_k}{\overset{\shortparallel}{e_k}} s_\mu\rangle = \begin{cases} 1 & \text{if } \lambda/\mu \text{ is a k-vert strip} \\ 0 & \text{else} \end{cases}$

(Pieri rule) $h_k^\perp(s_\lambda) = \sum_{\lambda/\mu \text{ is a k-hori strip}} s_\mu$ b/c $\langle h_k^\perp(s_\lambda), s_\mu\rangle = \langle s_\lambda, \underset{s_k}{\overset{\shortparallel}{h_k}} s_\mu\rangle = \begin{cases} 1 & \text{if } \lambda/\mu \text{ is a k-hori. strip} \\ 0 & \text{else} \end{cases}$

$p_k^\perp = k\dfrac{\partial}{\partial p_k}$)

• $\overline{M} = (1-q^{-1})(1-t^{-1})$

Recall: $B_\lambda(q,t) = \sum_{(i,j)\in\lambda} q^{i-1} t^{j-1}$

col row

• $f[\overline{B}]\tilde{H}_\mu = f[B_\mu(q^{-1},t^{-1})]\tilde{H}_\mu$

e.g.  $= (1+q+q^2)t^0 + (1+q)t^1 + (1+q)t^2 + t^3$

(1,1) (2,1) (3,1) ← row1 ← row2 ← row3 ← row4

Prop 3.3.1: There is an action of $\mathcal{E}$ on $\Lambda$ characterized as follows: ( → prop 3.3.3 (i)/(ii) to see why this characterizes the action of the full $\mathcal{E}$)

(i) $c_1 \mapsto 1$, $c_2 \mapsto (qt)^{-1}$  ($c_1, c_2$ acts as scalars)

(ii) $\Lambda_k(X^{1,0})$ and $\Lambda_k(X^{-1,0})$ acts as:

$$f(X^{1,0}) \mapsto (\omega f)\left[B - \tfrac{1}{M}\right], \quad f(X^{-1,0}) \mapsto (\omega f)\left[\overline{\tfrac{1}{M} - B}\right]$$

(iii) $\Lambda_k(X^{0,1})$ and $\Lambda_k(X^{0,-1})$ acts as:

$$f(X^{0,1}) \mapsto f\left[-\tfrac{X}{M}\right]^\bullet, \quad f(X^{0,-1}) \mapsto f(X)^\perp$$

Note: If $c_1^m c_2^n = 1$, then Heisenberg relation degenerates and $\Lambda(X^{\pm(m,n)})$ commutes.

Hence this action with $c_1 \mapsto 1$ makes $\Lambda(X^{\pm1,0})$ commutes, consistent with (ii)

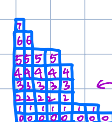
Also, $c_2 \mapsto \frac{1}{qt}$ makes Heisenberg relations of $\Lambda(X^{0,\pm1})$ consistent with (iii).

(See Shuffle Prop 3.3.1)

We focus on the use of operators representing the action of $p_1(X^{1,a})$ and $p_1(X^{a,1})$ in $\mathcal{E}$ on $\Lambda$.

(1) $\nabla$ operators:

Recall: $\nabla\tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{H}_\mu$  $\quad n(\mu) = \sum_{i=1}^{\ell(\mu)} (i-1)\mu_i$

 ← $n(\mu)$ = sum of the entries.

e.g. $n(87555421) = 1\times7 + 2\times5 + 3\times5 + 4\times5 + 5\times4 + 6\times2 + 7 = 91$

(2) Doubly infinite generating series:

$$D(z) := (\omega\Omega[z^{-1}X])^\bullet(\omega\Omega[-zMX])^\perp \quad \left(\text{or } D(-z) = \Omega[-z^{-1}X]^\bullet\,\Omega[zMX]^\perp\right)$$

Note: $\langle\Omega[AX], f(X)\rangle = \langle \sum s_\lambda[A]s_\lambda[X], f(X)\rangle = f[A] \Rightarrow \langle\Omega[AX]\Omega[BX], f(X)\rangle = \langle\Omega[(A+B)X], f(X)\rangle = f[A+B] = \langle\Omega[BX], f[X+A]\rangle$

$\Rightarrow (\Omega[AX])^\perp f(X) = f[X+A]$  (Take $A = z$, $f = p_k$. Then $\exp\left(\sum_{m>0} \frac{p_m^\perp z^m}{m}\right) p_k(X) = p_k[X+z] = p_k(X) + z^k$)

b/c B is arbitrary. verifies $p_m^\perp = m\frac{\partial}{\partial p_m}$

$\exp\left(\sum_{m>0} z^m \frac{\partial}{\partial p_m}\right) p_k(X) = \left[\left(1 + z\frac{\partial}{\partial p_1} + z^2\left(\frac{\partial}{\partial p_1}\right)^2 + \cdots\right)\left(1 + z^2\frac{\partial}{\partial p_2} + z^4\left(\frac{\partial}{\partial p_2}\right)^2 + \cdots\right)\cdots\right] p_k(X) = p_k(X) + z^k$

$z^k\frac{\partial}{\partial p_k}$ cont. to 1

$\Rightarrow (\Omega[AX])^\perp\,\Omega[BX]^\bullet = \Omega[AB]\,\Omega[BX]^\bullet\,\Omega[AX]^\perp$

$(\Omega[AX])^\perp\,\Omega[BX]^\bullet f(X)$
$= (\Omega[AX])^\perp(\Omega[BX]f(X))$
$= \Omega[B(X+A)]f[X+A]$

$\Omega[AB]\,\Omega[BX]^\bullet\,\Omega[AX]^\perp f(X)$
$= \Omega[AB]\,\Omega[BX]\,f[X+A]$
$= \Omega[B(X+A)]\,f[X+A]$

* (Path: Lemma 3.4.1) $\nabla f(X^{m,n})\nabla^{-1} = f(X^{m+n,n})$ (See last 2 pages for proof)

Def: For $a \in \mathbb{Z}$, define operators on $\Lambda_k(X)$

- $E_a := \nabla^a e_1(X)^\bullet \nabla^{-a}$
- $D_a := \langle z^{-a} \rangle D(z)$ (i.e. $D(z) = \sum_{a \in \mathbb{Z}} D_a z^{-a}$)

Proof on the next page

* (Path: Lemma 3.3.3) $[(\omega p_k[-\frac{X}{M}])^\bullet, D_a] = -D_{a+k}$, $[(\omega p_k(X))^\perp, D_a] = D_{a-k}$

Prop 3.3.3: In the action of $\mathcal{E}$ on $\Lambda_k(X)$ given by Prop. 3.3.1:

(i) $p_1[-MX^{1,a}] = -Mp_1(X^{1,a}) \in \mathcal{E}$ acts as $D_a$;
(ii) $p_1[-MX^{a,1}] = -Mp_1(X^{a,1}) \in \mathcal{E}$ acts as $E_a$.

Hence $D_1 = E_1 = p_1[-MX^{1,1}]$

Proof: (i) Known: $D_0 \tilde{H}_\mu = (1 - MB_\mu)\tilde{H}_\mu$

By Prop 3.3.1 (ii), $p_1[-MX^{1,0}] \mapsto (\omega p_1)[-M(B - \frac{1}{M})] = 1 - MB$ (= action of $D_0$)

Recall $[\omega p_k(X^{1,0}), p_1(X^{a,1})] = p_1(X^{a+k,1})$, $[\omega p_k(X^{-1,0})\ p_1(X^{a,1})] = -p_1(X^{a-k,1})$

Apply the anti-involution (reverse multiplication) $\Phi$, we have

$$-[\omega p_k(X^{0,1}), p_1(X^{1,a})] = p_1(X^{1,a+k}),\quad -[\omega p_k(X^{0,-1}_{\frac{\pi}{2}-\pi}), p_1(X^{1,a})] = -p_1(X^{1,a-k})$$

$\in (-\pi, \pi]$

Hence we have:

$$[\omega p_k(X^{0,1}), p_1(X^{1,a})] = -p_1(X^{1,a+k}),\quad [\omega p_k(X^{0,-1}), p_1(X^{1,a})] = p_1(X^{1,a-k})$$

$$\therefore\ [\underbrace{\omega p_k(X^{0,1})}_{(\omega p_k[-\frac{X}{M}])^\bullet}, p_1[-MX^{1,a}]] = -p_1[-MX^{1,a+k}],\quad [\underbrace{\omega p_k(X^{0,-1})}_{(\omega p_k(X))^\perp}, p_1[-MX^{1,a}]] = p_1[-MX^{1,a-k}]$$

By Prop 3.3.1 (ii)   By Prop 3.3.1 (iii)

compare

(ii) By Prop 3.3.1 (iii), $p_1[-MX^{0,1}]$ acts on $\Lambda_k(X)$ as multiplication by $p_1[-\frac{-MX}{M}] = p_1[X] = e_1(X)$. ($\therefore$ $a=0$ case holds)

By Lemma 3.4.1 in path, $\nabla f(X^{m,n}) \nabla^{-1} = f(X^{m+n,n}) \Rightarrow \nabla^a f(X^{m,n}) \nabla^{-a} = f(X^{m+an,n})$ $\forall a \in \mathbb{Z}$

$\therefore$ $p_1[-MX^{a,1}]$ acts as $\nabla^a p_1[-MX^{0,1}] \nabla^{-a} = \nabla^a e_1(X)^\bullet \nabla^{-a}$. $\square$

(Path: Lemma 3.3.3) $[(\omega p_k(-\frac{X}{M}))^{\bullet}, D_a] = -D_{a+k}$, $[(\omega p_k(X))^{\perp}, D_a] = D_{a-k}$.

Proof: First note that $[(\omega p_k(X))^{\perp}, D(z)] = [(\omega p_k(X))^{\perp}, \sum_{a\in\mathbb{Z}} D_a z^{-a}] = \sum_{a\in\mathbb{Z}} [(\omega p_k(X))^{\perp}, D_a] z^{-a}$

$\therefore [(\omega p_k(X))^{\perp}, D_a] = D_{a-k} \Leftrightarrow [(\omega p_k(X))^{\perp}, D(z)] = \sum_{a\in\mathbb{Z}} D_{a-k} z^{-a} = z^{-k} \sum_{a\in\mathbb{Z}} D_{a-k} z^{-(a-k)} = z^{-k} D(z)$

$\Leftrightarrow [(\omega p_k(X))^{\perp}, (\omega\Omega[z^{-1}X])^{\bullet} (\omega\Omega[-zMX])^{\perp}] = z^{-k} D(z)$

$\Leftrightarrow (\omega p_k(X))^{\perp} (\omega\Omega[z^{-1}X])^{\bullet} (\omega\Omega[-zMX])^{\perp} - (\omega\Omega[z^{-1}X])^{\bullet} (\omega\Omega[-zMX])^{\perp} (\omega p_k(X))^{\perp} = z^{-k} D(z)$

we can swap these

All operators of the form $f(X)^{\perp}$ commute with each other

$\Leftrightarrow \left((\omega p_k(X))^{\perp} (\omega\Omega[z^{-1}X])^{\bullet} - (\omega\Omega[z^{-1}X])^{\bullet} (\omega p_k(X))^{\perp}\right) \omega\Omega[-zMX]^{\perp} = z^{-k} D(z)$

$\Leftrightarrow [(\omega p_k(X))^{\perp}, (\omega\Omega[z^{-1}X])^{\bullet}] \cancel{(\omega\Omega[-zMX])^{\perp}} = z^{-k} (\omega\Omega[z^{-1}X])^{\bullet} \cancel{(\omega\Omega[-zMX])^{\perp}}$

$\Leftrightarrow [(\omega p_k(X))^{\perp}, (\omega\Omega[z^{-1}X])^{\bullet}] = z^{-k} (\omega\Omega[z^{-1}X])^{\bullet}$

$\Leftrightarrow [\omega \cdot p_k(X)^{\perp} \cdot \omega, \omega \cdot (\Omega[z^{-1}X])^{\bullet} \omega] = z^{-k} \omega \cdot (\Omega[z^{-1}X])^{\bullet} \omega$

Not affected by $\omega$

$\Leftrightarrow \cancel{\omega} (p_k(X)^{\perp} \Omega[z^{-1}X]^{\bullet} - \Omega[z^{-1}X]^{\bullet} p_k(X)^{\perp}) \cancel{\omega} = z^{-k} \cancel{\omega} (\Omega[z^{-1}X])^{\bullet} \cancel{\omega}$

$\Leftrightarrow [p_k(X)^{\perp}, \Omega[z^{-1}X]^{\bullet}] = z^{-k} \Omega[z^{-1}X]^{\bullet}$

$\Leftrightarrow [p_k(X)^{\perp}, \exp(\sum_{m>0} \frac{p_m(X) z^{-m}}{m})^{\bullet}] = z^{-k} \exp(\sum_{m>0} \frac{p_m(X) z^{-m}}{m})^{\bullet}$

which holds using the fact that $p_k(X)^{\perp} = k\frac{\partial}{\partial p_k}$.

Check: $p_k(X)^{\perp} \exp(\sum_{m>0} \frac{p_m(X) z^{-m}}{m})^{\bullet} f(X) - \exp(\sum_{m>0} \frac{p_m(X) z^{-m}}{m})^{\bullet} p_k(X)^{\perp} f(X)$

$= k\frac{\partial}{\partial p_k}\left(\exp(\sum_{m>0} \frac{p_m(X) z^{-m}}{m}) f(X)\right) - \exp(\sum_{m>0} \frac{p_m(X) z^{-m}}{m}) k\frac{\partial}{\partial p_k} f(X)$

$= \cancel{k} \exp(\sum_{m>0} \frac{p_m(X) z^{-m}}{m}) \cdot \frac{z^{-k}}{\cancel{k}} \cdot f(X) + \cancel{\exp(\sum_{m>0} \frac{p_m(X) z^{-m}}{m}) \cdot k\frac{\partial}{\partial p_k} f(X) - \exp(\sum_{m>0} \frac{p_m(X) z^{-m}}{m}) k\frac{\partial f(X)}{\partial p_k}}$

$= z^{-k} \exp(\sum_{m>0} \frac{p_m(X) z^{-m}}{m})^{\bullet} f(X)$

$\therefore [p_k(X)^{\perp}, \Omega[z^{-1}X]^{\bullet}] = z^{-k} \Omega[z^{-1}X]^{\bullet}$

Define a modified inner product:

$\langle g, h \rangle' := \langle g[-MX], h \rangle$

Note that • $\langle g[-MX], h \rangle = \langle g, h[-MX] \rangle$ because $\{p_\lambda\}$ is an orthogonal basis. $\therefore \langle , \rangle'$ is symmetric.

• For any $f \in \Lambda$, $f^{\perp}$ and $f[-\frac{X}{M}]^{\bullet}$ are adjoint w.r.t. $\langle , \rangle'$.

$\langle f[-\frac{X}{M}]^{\bullet} g, h \rangle' = \langle (f[-\frac{X}{M}] g)[-MX], h \rangle$
$= \langle f[\frac{-MX}{-M}] g[-MX], h \rangle$
$= \langle f(X) g[-MX], h \rangle$
$= \langle g[-MX], f^{\perp}(h) \rangle$
$= \langle g, f^{\perp}(h) \rangle'$

$\because D(z) = (\omega\Omega[z^{-1}X])^{\bullet} (\omega\Omega[-zMX])^{\perp}$ and $D(z^{-1}) = (\omega\Omega[zX])^{\bullet} (\omega\Omega[-z^{-1}MX])^{\perp}$

$\langle D(z) g, h \rangle' = \langle (\omega\Omega[z^{-1}X])^{\bullet} (\omega\Omega[-zMX])^{\perp} g, h \rangle'$

$= \langle (\omega\Omega[-zMX])^{\perp} g, (\omega\Omega[-Mz^{-1}X])^{\perp} h \rangle'$

$= \langle g, (\omega\Omega[zX])^{\bullet} (\omega\Omega[-z^{-1}MX])^{\perp} h \rangle' = \langle g, D(z^{-1}) h \rangle'$

$\therefore$ $D(z^{-1})$ is the $\langle\, ,\, \rangle'$-adjoint of $D(z)$.

Hence $D_{-a}$ is the $\langle\, ,\, \rangle'$-adjoint of $D_a$.

Taking adjoint on both sides of $[(\omega p_k(X))^{\perp}, D_a] = D_{a-k}$, we have

$$[(\omega p_k(X))^{\bullet}, D_{-a}] = D_{-a+k}$$

Hence $[(\omega p_k(X))^{\bullet}, D_a] = D_{a+k}$.

(Path: Lemma 3.4.1) $\nabla f(X^{m,n}) \nabla^{-1} = f(X^{m+n,n})$.

Proof: We use the following facts:

① $\nabla \cdot p_1(X) \cdot \nabla^{-1} = D_1$, $\nabla^{-1} p_1(X)^{\perp} \cdot \nabla = -\frac{1}{M} D_{-1}$ (by Bergeron, Garsia, Haiman, Tesler 1999. c.f. [3] in 'Path' reference)

② The group of k-algebra automorphisms of $\mathcal{E}$ includes one which acts on $\Lambda_k(X^{m,n})$ by $f(X^{m,n}) \mapsto f(X^{m+n,n})$, and on $F = k[c_1^{\pm1}, c_2^{\pm1}]$ by an automorphism which fixes the central character in Prop. 3.3.1 (i).

For $m = \pm1$, $n = 0$: By Prop 3.3.1 (ii), $f(X^{\pm1,0})$ acts on $\Lambda$ as Macdonald eigenoperators and hence commutes with $\nabla$

$\therefore$ $\nabla \cdot f(X^{\pm1,0}) = f(X^{\pm1,0}) \cdot \nabla \Leftrightarrow \nabla f(X^{\pm1,0}) \nabla^{-1} = f(X^{\pm1,0})$

For $n > 0$ : $\mathcal{E}^{*,>0}$ is generated by $\{p_1(X^{a,1}) : a \in \mathbb{Z}\}$. Hence it suffices to check $\nabla p_1(X^{a,1}) \nabla^{-1} = p_1(X^{a+1,1})$.

Recall the useful relations :

$$[\omega p_k(X^{1,0}), p_1(X^{a,1})] = p_1(X^{a+k,1}), \quad [\omega p_k(X^{-1,0}), p_1(X^{a,1})] = -p_1(X^{a-k,1}).$$

Suppose we can prove $\nabla p_1(X^{0,1}) \nabla^{-1} = p_1(X^{1,1})$

$\because$ $\nabla$ commutes with action of $\Lambda(X^{\pm1,0})$

$$\begin{aligned} \therefore [\omega p_k(X^{1,0}), \nabla p_1(X^{0,1}) \nabla^{-1}] &= \omega p_k(X^{1,0}) \nabla p_1(X^{0,1}) \nabla^{-1} - \nabla p_1(X^{0,1}) \nabla^{-1} \omega p_k(X^{1,0}) \\ &= \nabla \omega p_k(X^{1,0}) p_1(X^{0,1}) \nabla^{-1} - \nabla p_1(X^{0,1}) \omega p_k(X^{1,0}) \nabla^{-1} \\ &= \nabla [\omega p_k(X^{1,0}), p_1(X^{0,1})] \nabla^{-1} \\ &= \nabla p_1(X^{k,1}) \nabla^{-1} \end{aligned}$$

Assume this is true

i.e. $[\omega p_k(X^{1,0}), p_1(X^{1,1})] = \nabla p_1(X^{k,1}) \nabla^{-1}$

$\therefore$ $p_1(X^{1+k,1}) = \nabla p_1(X^{k,1}) \nabla^{-1} \Rightarrow \nabla p_1(X^{a,1}) \nabla^{-1} = p_1(X^{a+1,1})$ for $a \geq 0$

Similarly, $[\omega p_k(X^{-1,0}), \nabla p_1(X^{0,1}) \nabla^{-1}] = -\nabla p_1(X^{-k,1}) \nabla^{-1}$

assume this is true

i.e. $[\omega p_k(X^{-1,0}), p_1(X^{1,1})] = -\nabla p_1(X^{-k,1}) \nabla^{-1}$

$$\cancel{-} p_1(X^{1-k,1}) = \cancel{-} \nabla p_1(X^{-k,1}) \nabla^{-1}$$

Put $a = -k$, we have $\nabla p_1(X^{a,1}) \nabla^{-1} = p_1(X^{a+1,1})$ for $a \leq 0$.

Thus, we only need to check if $\nabla p_1(X^{0,1})\nabla^{-1} = p_1(X^{1,1})$

By Prop 3.3.1 (iii), $p_1[-MX^{0,1}]$ acts as $p_1\left[\frac{-MX}{-M}\right]^\bullet = p_1(X)^\bullet$.

By Prop 3.3.3 (i), $p_1[-MX^{1,1}]$ acts as $D_1$.

proof independent of this Lemma

$\therefore$ $\nabla p_1[-MX^{0,1}]\nabla^{-1}$ acts as $\nabla p_1(X)^\bullet\nabla^{-1}$ which is $D_1$ by fact ②.

i.e. $\nabla p_1[-MX^{0,1}]\nabla^{-1} = p_1[-MX^{1,1}]$

$\therefore$ $-M(\nabla p_1(X^{0,1})\nabla^{-1}) = -Mp_1(X^{1,1}) \Leftrightarrow \nabla p_1(X^{0,1})\nabla^{-1} = p_1(X^{1,1})$ which completes the proof for $\mathcal{E}^{*,>0}$.

For $n<0$, we start with

$$[\omega p_k(X^{1,0}), p_1(X^{a,-1})] = -p_1(X^{a+k,-1}), \quad [\omega p_k(X^{-1,0}), p_1(X^{a,-1})] = p_1(X^{a-k,-1}).$$

Suppose we can prove that $\nabla p_1(X^{1,-1})\nabla^{-1} = p_1(X^{0,-1})$

Then $[\omega p_k(X^{1,0}), \nabla p_1(X^{1,-1})\nabla^{-1}] = \nabla[\omega p_k(X^{1,0}), p_1(X^{1,-1})]\nabla^{-1} = -\nabla p_1(X^{1+k,-1})\nabla^{-1}$

$\therefore$ $[\omega p_k(X^{1,0}), p_1(X^{0,-1})] = -\nabla p_1(X^{k+1,-1})\nabla^{-1}$

$\therefore$ $\cancel{-}p_1(X^{k,-1}) = \cancel{-}\nabla p_1(X^{k+1,-1})\nabla^{-1}$

Put $k=a-1$, we have

$$\nabla p_1(X^{a,-1})\nabla^{-1} = p_1(X^{a-1,-1}) \quad \forall a \geq 1$$

Similarly, $[\omega p_k(X^{-1,0}), \nabla p_1(X^{1,-1})\nabla^{-1}] = \nabla[\omega p_k(X^{-1,0}), p_1(X^{1,-1})]\nabla^{-1} = \nabla p_1(X^{1-k,-1})\nabla^{-1}$

$\Rightarrow$ $[\omega p_k(X^{-1,0}), p_1(X^{0,-1})] = \nabla p_1(X^{1-k,-1})\nabla^{-1}$

$\Leftrightarrow$ $p_1(X^{-k,-1}) = \nabla p_1(X^{1-k,-1})\nabla^{-1}$

Put $k=1-a$, we get $\nabla p_1(X^{a,-1})\nabla^{-1} = p_1(X^{a-1,-1}) \quad \forall a \leq 1$

$\therefore$ It remains to prove $\nabla p_1(X^{1,-1})\nabla^{-1} = p_1(X^{0,-1})$

By Prop 3.3.1 (iii), $p_1(X^{0,-1})$ acts as $p_1(X)^\perp$
By Prop 3.3.3 (i), $p_1[-MX^{1,-1}]$ acts as $D_{-1}$

$\therefore$ $\nabla p_1[-MX^{1,-1}]\nabla^{-1}$ acts as $\nabla D_{-1}\nabla^{-1} = -Mp_1(X)^\perp$

i.e. $\nabla p_1[-MX^{1,-1}]\nabla^{-1} = -Mp_1(X^{0,-1})$

$\therefore$ $\nabla p_1(X^{1,-1})\nabla^{-1} = p_1(X^{0,-1})$ and result follows. $\square$