- GL_ℓ characters
  - $\Bbbk$: a field containing $\mathbb{Q}(q)$
  - Weight lattice : $X = \mathbb{Z}^\ell$
  - Weyl group : $W = S_\ell$
  - positive roots : $\varepsilon_i - \varepsilon_j$ for $1 \le i < j \le \ell$
  - simple roots : $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \le i \le \ell-1$
  - $(\ ,\ ) : \mathbb{Z}^\ell \times \mathbb{Z}^\ell \to \mathbb{Z}$ s.t. $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$

    Hence coroots and roots coincide and $\alpha_i^\vee = \alpha_i$ $\forall\ 1 \le i \le \ell-1$.
  - $(\lambda_1, \dots, \lambda_\ell) \in \mathbb{Z}^\ell$: dominant if $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_\ell$

    regular if $\lambda_i \ne \lambda_j$ $\forall i \ne j$

    has trivial stabilizer in S_ℓ
  - Polynomial weight : dominant + $\lambda_\ell \ge 0$. (i.e. Partition)
  - algebra of virtual GL_ℓ-char : $(\Bbbk X)^W$: can be identified with $\underbrace{\Bbbk[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_\ell^{\pm 1}]^{S_\ell}}_{\text{algebra of symmetric Laurent polynomials}}$.

    For a polynomial weight λ, the irreducible character χ_λ is s_λ.

    Given a virtual GL_ℓ-character $f(x) = f(x_1, \dots, x_\ell) = \sum_\lambda c_\lambda \chi_\lambda$, denote the partial sum over polynomial weights λ by $f(x)_{pol}$ (hence $\in \Lambda(x_1, \dots, x_\ell)$)

    (When $f(x)$ is an infinite formal sum of irreducible GL_ℓ-characters, $f(x)_{pol}$ is a symmetric formal power series)
  - Weyl symmetrization operator for GL_ℓ :

$$\sigma\, f(x_1, \dots, x_\ell) := \sum_{w \in S_\ell} w\left(\frac{f(x)}{\prod_{i<j}\left(1 - \frac{x_j}{x_i}\right)}\right)$$

e.g.

$$\sigma\, f(x_1, x_2, x_3) = \sum_{w \in S_\ell} w\left(\frac{f(x_1, x_2, x_3)}{\left(1 - \frac{x_2}{x_1}\right)\left(1 - \frac{x_3}{x_1}\right)\left(1 - \frac{x_3}{x_2}\right)}\right)$$

$$= \sum_{w \in S_\ell} w \frac{x_1^2 x_2 f(x_1, x_2, x_3)}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}$$

$$= \frac{x_1^2 x_2 f(x_1, x_2, x_3)}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)} + \frac{x_1 x_2^2 f(x_2, x_1, x_3)}{(x_2 - x_1)(x_2 - x_3)(x_1 - x_3)} + \frac{x_2 x_3^2 f(x_3, x_2, x_1)}{(x_3 - x_2)(x_3 - x_1)(x_2 - x_1)} + \frac{x_1^2 x_3 f(x_1, x_3, x_2)}{(x_1 - x_3)(x_1 - x_2)(x_2 - x_3)} + \frac{x_2^2 x_3 f(x_2, x_3, x_1)}{(x_2 - x_3)(x_2 - x_1)(x_3 - x_1)} + \frac{x_1 x_3^2 f(x_3, x_1, x_2)}{(x_3 - x_1)(x_3 - x_2)(x_1 - x_2)}$$

$$= \frac{1}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}\left[x_1^2 x_2 f(x_1, x_2, x_3) - x_1 x_2^2 f(x_2, x_1, x_3) - x_2 x_3^2 f(x_3, x_2, x_1) - x_1^2 x_3 f(x_1, x_2, x_3) + x_2^2 x_3 f(x_2, x_3, x_1) + x_1 x_3^2 f(x_3, x_1, x_2)\right]$$

  - Weyl character formula : $\chi_\lambda = \sigma(x^\lambda)$ for dominant weights λ (In general, if $\varphi(x) = \varphi(x_1, \dots, x_\ell)$ is a Laurent polynomial over any field $\Bbbk$, then $\sigma(\varphi(x))$ is a virtual GL_ℓ-character over $\Bbbk$.)

e.g.

$$\chi_{532} = \sigma(x^{532}) = \sum_{w \in S_3} w\left(\frac{x_1^2 x_2 \cdot x_1^5 x_2^3 x_3^2}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}\right)$$

$$= \frac{x_1^2 x_2^2 x_3^2}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}\left(x_1^5 x_2^2 - x_1^2 x_2^5 - x_2^2 x_3^5 - x_1^5 x_3^2 + x_2^5 x_3^2 + x_1^2 x_3^5\right)$$

$$= x_1^2 x_2^2 x_3^2 \left(x_1^3 x_2 + x_1^3 x_3 + x_2^3 x_3 + x_1 x_2^3 + x_1 x_3^3 + x_2 x_3^3 + x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + 2 x_1^2 x_2 x_3 + 2 x_1 x_2^2 x_3 + 2 x_1 x_2 x_3^2\right)$$

  - Fix a weight ρ s.t. $\langle \alpha_i^\vee, \rho \rangle = 1$ $\forall$ simple coroot $\alpha_i^\vee$ (e.g. $\rho = (\ell-1, \ell-2, \dots, 1, 0)$) ($\rho$: unique up to adding a constant vector)
    - If $\mu \in \mathbb{Z}^\ell$ s.t. $\mu + \rho$ is not regular (i.e. with equal parts), then $\sigma(x^\mu) = 0$.
    - If $\mu \in \mathbb{Z}^\ell$ s.t. $\mu + \rho$ is regular, then for $\varpi \in S_\ell$ s.t. $\varpi(\mu + \rho) - \rho$ is dominant, then $\sigma(x^\mu) = (-1)^{\ell(w)} \chi_{\varpi(\mu+\rho) - \rho}$

e.g. • $\mu = (1, 2, 0)$ and $\mu + (2, 1, 0) = (3, 3, 0)$ $\quad \sigma(x^\mu) = \sum_{w \in S_3} w\left(\frac{\overbrace{x_1^2 x_2}^{\text{That is why we have } +(2,1,0)} \cdot x_1^1 x_2^2 x_3^0}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}\right) = 0$

• $\mu = (1, 3, 0)$ and $\mu + (2, 1, 0) = (3, 4, 0)$

Then $\varpi = s_1$. Hence $\sigma(x^{1,3,0}) = \sum_{w \in S_3} w\left(\frac{x_1^2 x_2 \cdot x_1 x_2^3 x_3^0}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}\right) = \sum_{w \in S_3} w\left(\frac{x_1^3 x_2^4 x_3^0}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}\right) = \sum_{w \in S_3} (-1)\, w\left(\frac{x_1^4 x_2^3 x_3^0}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}\right) = (-1)\,\sigma(x^{2,2,0})$

Lemma 2.3.1 (Path): For any GL_ℓ weights λ, μ and Laurent polynomial $\phi(x) = \phi(x_1, x_2, \dots, x_\ell)$, we have

$$\text{(i)}\quad \overline{\chi_\lambda} \prod_{i<j}\left(1-\frac{x_i}{x_j}\right) = \sum_{w\in S_\ell} (-1)^{\ell(w)} x^{-w(\lambda+\rho)+\rho}$$

$$\text{(ii)}\quad \langle \chi_\lambda \rangle\, \sigma(\phi(x)) = \langle x^0 \rangle\, \overline{\chi_\lambda}\, \phi(x) \prod_{i<j}\left(1-\frac{x_i}{x_j}\right)$$

$$\text{(iii)}\quad \sigma(x^\mu)_{pol} = \langle z^{-\mu} \rangle\, \Omega[\overline{Z} X] \prod_{i<j}\left(1-\frac{z_i}{z_j}\right)$$

in alphabets $X = x_1 + \cdots + x_\ell$ and $Z = z_1 + \cdots + z_\ell$, $\overline{Z} = z_1^{-1} + \cdots + z_\ell^{-1}$.

Proof: (i) $\overline{\chi_\lambda} = \overline{\sigma(x^\lambda)} = \overline{\sum_{w\in S_\ell} w\left(\frac{x^\lambda}{\prod_{i<j}(1-\frac{x_j}{x_i})}\right)} = \sum_{w\in S_\ell} w\left(\frac{x^{-\lambda}}{\prod_{i<j}(1-\frac{x_i}{x_j})}\right) = \sum_{w\in S_\ell} \frac{(-1)^{\ell(w)} x^{-w(\lambda+\rho)+\rho}}{\prod_{i<j}(1-\frac{x_i}{x_j})}$ and (i) follows.

(ii) By linearity, it suffices to verify $\phi(x) = x^\mu$

$$\langle x^0\rangle \overline{\chi_\lambda}\, \phi(x) \prod_{i<j}\left(1-\frac{x_i}{x_j}\right) = \langle x^{-\mu}\rangle \overline{\chi_\lambda} \prod_{i<j}\left(1-\frac{x_i}{x_j}\right) \overset{\text{by (i)}}{=} \langle x^{-\mu}\rangle \sum_{w\in S_\ell} (-1)^{\ell(w)} x^{-w(\lambda+\rho)+\rho} = \begin{cases} (-1)^{\ell(w)} & \text{if } -\mu = -w(\lambda+\rho)+\rho \text{ for some } w\in S_\ell \\ 0 & \text{otherwise} \end{cases} = \begin{cases} (-1)^{\ell(w)} & \text{if } \mu+\rho = w(\lambda+\rho) \text{ for some } w\in S_\ell \\ 0 & \text{otherwise} \end{cases}$$

$$\because\ \sigma(x^\mu) = (-1)^{\ell(w)} \chi_{w(\mu+\rho)-\rho} \Rightarrow \langle \chi_\lambda\rangle \sigma(x^\mu) = \begin{cases} (-1)^{\ell(\gamma)} & \text{if } \lambda = \gamma(\mu+\rho)-\rho \text{ for some } \gamma\in S_\ell \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} (-1)^{\ell(\gamma^{-1})} & \text{if } \gamma^{-1}(\lambda+\rho) = \mu+\rho \text{ for some } \gamma\in S_\ell \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} (-1)^{\ell(w)} & \text{if } w(\lambda+\rho) = \mu+\rho \text{ for some } w\in S_\ell \\ 0 & \text{otherwise} \end{cases}$$

$\therefore\ \langle \chi_\lambda\rangle \sigma(x^\mu) = \langle x^0\rangle \overline{\chi_\lambda}\, x^\mu \prod_{i<j}\left(1-\frac{x_i}{x_j}\right)$ and (ii) follows by linearity.

(iii) $\langle z^{-\mu}\rangle \Omega[\overline{Z}X] \prod_{i<j}\left(1-\frac{z_i}{z_j}\right) \underset{\text{by Cauchy formula}}{=} \sum_\lambda s_\lambda(X)\cdot \langle z^{-\mu}\rangle s_\lambda[\overline{Z}] \prod_{i<j}\left(1-\frac{z_i}{z_j}\right) = \sum_\lambda s_\lambda(X)\cdot \langle \chi_\lambda\rangle \sigma(x^\mu) = \sigma(x^\mu)_{pol}$ $\square$

- Hall-Littlewood symmetrization

$\phi(x)$: Laurent polynomial over a field containing $\mathbb{Q}(q)$

$\frac{1}{1-q\frac{x_i}{x_j}}$: power series in q $\Rightarrow H_q^\ell(\phi(x))$: formal Laurent series in q over virtual GL_ℓ characters if we expand the coeff of $\phi(x)$ as formal Laurent series in q

indicate # variables

$$H_q^\ell(\phi(x)) = \sigma\left(\frac{\phi(x)}{\prod_{i<j}(1-q\frac{x_i}{x_j})}\right) = \sum_{w\in S_\ell} w\left(\frac{\phi(x)}{\prod_{i<j}\left((1-\frac{x_j}{x_i})\underbrace{(1-q\frac{x_i}{x_j})}_{\text{Geom. series}}\right)}\right).$$

$\leftarrow$ infinite formal sum of irreducible GL_ℓ characters with coeff. in $\mathbb{Q}(q)$

$*$ $H_q^\ell(x^\mu)_{pol} = H_\mu(x_1, \dots, x_\ell; q) = \sum_\lambda \overbrace{K_{\lambda,\mu}(q)}^{q\text{-Kostka coeff.}} s_\lambda$ (dual Hall-Littlewood polynomials in ℓ variables)

e.g. $H_q^2(x^{53}) = \sum_{w\in S_2} w\left(\frac{x^{53}}{(1-\frac{x_2}{x_1})(1-q\frac{x_1}{x_2})}\right)$

$$= \frac{x_1^5 x_2^3}{(1-\frac{x_2}{x_1})(1-q\frac{x_1}{x_2})} + \frac{x_1^3 x_2^5}{(1-\frac{x_1}{x_2})(1-q\frac{x_2}{x_1})}$$

$$= \frac{x_1^6 x_2^3}{x_1-x_2}\left(1+q\frac{x_1}{x_2}+q^2\frac{x_1^2}{x_2^2}+\cdots\right) + \frac{x_1^3 x_2^6}{x_2-x_1}\left(1+q\frac{x_2}{x_1}+q^2\frac{x_2^2}{x_1^2}+\cdots\right)$$

$$= \frac{x_1^3 x_2^3}{x_1-x_2}\left[(x_1^3-x_2^3) + q\left(\frac{x_1^4}{x_2}-\frac{x_2^4}{x_1}\right) + q^2\left(\frac{x_1^5}{x_2^2}-\frac{x_2^5}{x_1^2}\right) + \cdots\right]$$

$$= \overbrace{x_1^3x_2^3(x_1^2+x_1x_2+x_2^2)}^{\chi_{53}} + q\overbrace{x_1^2x_2^2(x_1^4+x_1^3x_2+x_1^2x_2^2+x_1x_2^3+x_2^4)}^{\chi_{62}} + q^2\overbrace{x_1x_2(x_1^6+x_1^5x_2+\cdots+x_1x_2^5+x_2^6)}^{\chi_{71}} + q^3\overbrace{(x_1^8+x_1^7x_2+\cdots+x_1x_2^7+x_2^8)}^{\chi_{80}}$$
$$+ q^4\underbrace{\left(\frac{x_1^{10}+x_1^9x_2+x_1^8x_2^2+\cdots+x_1x_2^9+x_2^{10}}{x_1x_2}\right)}_{\chi_{9,-1}} + q^5\underbrace{\left(\frac{x_1^{12}+x_1^{11}x_2+x_1^{10}x_2^2+\cdots+x_1^2x_2^{10}+x_1x_2^{11}+x_2^{12}}{x_1^2x_2^2}\right)}_{\chi_{10,-2}} + \cdots\cdots$$

$\therefore\ H_q^2(x^{53})_{pol} = \chi_{53} + q\chi_{62} + q^2\chi_{71} + q^3\chi_8 = s_{53} + qs_{62} + q^2s_{71} + q^3s_8$. "raising operator series" sense

$*$ If $\phi(x_1, \dots, x_\ell)$ is a rational function over a field k containing $\mathbb{Q}(q)$, then $H_q^\ell(\phi(x))$ is a symmetric rational function over k.

$\mathbb{Q}(q,t)$ version:

$$H^{\ell}_{q,t}(\phi(x)) := H^{\ell}_{q}\left(\phi(x)\prod_{i<j}\frac{1-qt\frac{x_i}{x_j}}{1-t\frac{x_i}{x_j}}\right) = \sigma\left(\phi(x)\prod_{i<j}\frac{1-qt\frac{x_i}{x_j}}{(1-q\frac{x_i}{x_j})(1-t\frac{x_i}{x_j})}\right) = \sum_{w\in S_\ell} w\left(\phi(x)\prod_{i<j}\frac{1-qt\frac{x_i}{x_j}}{(1-\frac{x_j}{x_i})(1-q\frac{x_i}{x_j})(1-t\frac{x_i}{x_j})}\right)$$

e.g. $$H^{2}_{q,t}(x^{53}) = \sum_{w\in S_2} w\left(\frac{x_1^5x_2^3(1-qt\frac{x_1}{x_2})}{(1-\frac{x_2}{x_1})(1-q\frac{x_1}{x_2})(1-t\frac{x_1}{x_2})}\right)$$

$$= \frac{x_1^5x_2^3(1-qt\frac{x_1}{x_2})}{(1-\frac{x_2}{x_1})(1-q\frac{x_1}{x_2})(1-t\frac{x_1}{x_2})} + \frac{x_1^3x_2^5(1-qt\frac{x_2}{x_1})}{(1-\frac{x_1}{x_2})(1-q\frac{x_2}{x_1})(1-t\frac{x_2}{x_1})}$$

$$= \frac{x_1^6x_2^3}{(x_1-x_2)}\left(1+q\frac{x_1}{x_2}+q^2\frac{x_1^2}{x_2^2}+\cdots\right)\left(1+t\frac{x_1}{x_2}+t^2\frac{x_1^2}{x_2^2}+\cdots\right)\left(1-qt\frac{x_1}{x_2}\right) - \frac{x_1^3x_2^6}{(x_1-x_2)}\left(1+q\frac{x_2}{x_1}+q^2\frac{x_2^2}{x_1^2}+\cdots\right)\left(1+t\frac{x_2}{x_1}+t^2\frac{x_2^2}{x_1^2}+\cdots\right)\left(1-qt\frac{x_2}{x_1}\right)$$

$$= \frac{x_1^6x_2^3}{x_1-x_2}\left[1+(q+t)\frac{x_1}{x_2}+(q^2+qt+t^2)\frac{x_1^2}{x_2^2}+\cdots\right]\left(1-qt\frac{x_1}{x_2}\right) - \frac{x_1^3x_2^6}{x_1-x_2}\left[1+(q+t)\frac{x_2}{x_1}+(q^2+qt+t^2)\frac{x_2^2}{x_1^2}+\cdots\right]\left(1-qt\frac{x_2}{x_1}\right)$$

$$= \frac{x_1^3x_2^3}{x_1-x_2}\left[(x_1^3-x_2^3)+(q+t)\left(\frac{x_1^4}{x_2}-\frac{x_2^4}{x_1}\right)+(q^2+qt+t^2)\left(\frac{x_1^5}{x_2^2}-\frac{x_2^5}{x_1^2}\right)+\cdots\right] - qt\frac{x_1^2x_2^2}{(x_1-x_2)}\left[(x_1^5-x_2^5)+(q+t)\left(\frac{x_1^6}{x_2}-\frac{x_2^6}{x_1}\right)+\cdots\right]$$

$$= x_1^3x_2^3(x_1^2+x_1x_2+x_2^2)+(q+t)(x_1^4+x_1^3x_2+x_1^2x_2^2+x_1x_2^3+x_2^4)+(q^2+qt+t^2)(x_1^6+x_1^5x_2+\cdots+x_1x_2^5+x_2^6)$$
$$+(q^3+q^2t+qt^2+t^3)(x_1^8+x_1^7x_2+\cdots+x_1x_2^7+x_2^8)+(q^4+q^3t+q^2t^2+qt^3+t^4)\left(\frac{x_1^{10}+x_1^9x_2+\cdots+x_1x_2^9+x_2^{10}}{x_1x_2}\right)+\cdots$$
$$-qt\,x_1^2x_2^2(x_1^4+x_1^3x_2+x_1^2x_2^2+x_1x_2^3+x_2^4)-qt(q+t)x_1^2x_2^2(x_1^6+x_1^5x_2+\cdots+x_1x_2^5+x_1^6)-\cdots$$

$$= \chi_{53}+(q+t)\chi_{62}+(q^2+qt+t^2)\chi_{71}+(q^3+q^2t+qt^2+t^3)\chi_{80}+(q^4+q^3t+q^2t^2+qt^3+t^4)\chi_{9,-1}+\cdots$$
$$-qt\,\chi_{62}-qt(q+t)\chi_{71}-qt(q^2+qt+t^2)\chi_{80}-qt(q^3+q^2t+qt^2+t^3)\chi_{9,-1}+\cdots$$

$$\therefore H^{2}_{q,t}(x^{53})_{pol} = \chi_{53}+(q+t-qt)\chi_{62}+(q^2+qt+t^2-q^2t-qt^2)\chi_{71}+(q^3+q^2t+qt^2+t^3-q^3t-q^2t^2-qt^3)\chi_{80}.$$

Fix $k=\mathbb{Q}(q,t)$ (and use z instead of x as variables)

Define $T := T(k[z,z^{-1}])$ be the tensor algebra on the Laurent polynomial ring in one variable (non-commutative polynomial algebra with generators corresponding to z^a of $k[z,z^{-1}]$ as a vector space)
(or $T^{\cdot}(k[z,z^{-1}])$) ← i.e. $\langle z^a : a\in\mathbb{Z}\rangle_k$

Identify $T^m = T^m(k[z^{\pm1}])$ with $k[z_1^{\pm1},z_2^{\pm1},\dots,z_m^{\pm1}]$ and the product in T is given by 'concatenation':

For $f\in T^m$, $g\in T^\ell$

$$f\cdot g := f(z_1,z_2,\dots,z_m)\,g(z_{m+1},z_{m+2},\dots,z_{m+\ell})$$

e.g. $T\otimes T := \{f(z)\otimes g(z) : f,g\in k[z,z^{-1}]\} = \mathrm{span}_k\{z^a\otimes z^b : a,b\in\mathbb{Z}\}$
(T^2)

$\updownarrow \qquad \updownarrow$

$k[z_1^{\pm1},z_2^{\pm1}] = \mathrm{span}_k\{z_1^az_2^b : a,b\in\mathbb{Z}\}$

Define $I^\ell := \ker H^\ell_{q,t} = \{\phi(x)\in k[z_1^{\pm1},\dots,z_\ell^{\pm1}] : H^\ell_{q,t}(\phi(x))=0\}\subseteq T^\ell$, $\quad I := \bigoplus_{\ell\in\mathbb{Z}_{\geq0}} I^\ell \subseteq T$ (Note: I is a graded two-sided ideal in T)

Fact: The Feign-Tsymbauliak shuffle algebra is the quotient $S := T/I$.

Prop 3.4.1 (Schiffmann, Vasserot 2013) There is an algebra isomorphism $\psi : S\to\mathcal{E}^+$, $z^a\mapsto p_1[-MX^{1,a}]$ and anti-isomorphism $\psi^{op} := \Phi\circ\psi : S\to\Phi\mathcal{E}^+$, $z^a\mapsto p_1[-MX^{a,1}]$

* On monomials in m variables, representing elements of tensor degree m in S,

$$\psi(z_1^{a_1} z_2^{a_2} \cdots z_m^{a_m}) = p_1[-MX^{1,a_1}] \cdots p_1[-MX^{1,a_m}]$$

$$\psi^{op}(z_1^{a_1} z_2^{a_2} \cdots z_m^{a_m}) = p_1[-MX^{a_m,1}] \cdots p_1[-MX^{a_1,1}]$$

Prop 3.4.2 (Path Prop 3.5.2) Let $\phi(z) = \phi(z_1, z_2, \ldots, z_\ell)$ be a Laurent polynomial representing an element of tensor degree ℓ in S.

Let $\zeta = \psi(\phi(z)) \in \mathcal{E}^+$.

With $\mathcal{E}$ acting on Λ as in Prop 3.3.1, we have

$$\omega(\zeta \cdot 1)(x_1, x_2, \ldots, x_\ell) = H^{\ell}_{q,t}(\phi(x))_{pol}.$$

Proof: By linearity, it suffices to check $\phi(z) = z_1^{a_1} \cdots z_\ell^{a_\ell}$. In this case $\zeta = \psi(\phi(z)) = \psi(z_1^{a_1} \cdots z_\ell^{a_\ell}) = p_1[-MX^{1,a_1}]\, p_1[-MX^{1,a_2}] \cdots p_1[-MX^{1,a_\ell}]$

Hence by Prop 3.3.3, ζ acts on Λ as $D_{a_1} D_{a_2} \cdots D_{a_\ell}$.

switch with a factor

Recall $(\Omega[AX])^{\perp}\, \Omega[BX]^{\bullet} = \Omega[AB]\, \Omega[BX]^{\bullet}\, \Omega[AX]^{\perp}$ (from §3.3)

Apply ω: $(\omega\Omega[AX])^{\perp}(\omega\Omega[BX]^{\bullet}) = \Omega[AB]\,(\omega\Omega[BX])^{\bullet}(\omega\Omega[AX])^{\perp}$

No "ω" b/c it is a constant (ie independent of X)

$$D(z_1) D(z_2) \cdots D(z_\ell) = (\omega\Omega[z_1^{-1}X])^{\bullet} (\omega\Omega[-z_1MX])^{\perp} (\omega\Omega[z_2^{-1}X])^{\bullet} (\omega\Omega[-z_2MX])^{\perp} \cdots (\omega\Omega[z_\ell^{-1}X])^{\bullet} (\omega\Omega[-z_\ell MX])^{\perp}$$

factors created: $\Omega[-z_i M z_j^{-1}]$ $\forall\, i<j$

{ group ( )$^{\bullet}$ in front & group ( )$^{\perp}$ in the end

$$= \prod_{1\le i<j\le \ell} \Omega[-z_i M z_j^{-1}]\, (\omega\Omega[(z_1^{-1}+z_2^{-1}+\cdots+z_\ell^{-1})X])^{\bullet} (\omega\Omega[-(z_1+\cdots+z_\ell)MX])^{\perp}$$

e.g. switch

$$(\omega\Omega[z_1^{-1}X])^{\bullet} (\omega\Omega[-z_1MX])^{\perp} (\omega\Omega[z_2^{-1}X])^{\bullet} (\omega\Omega[-z_2MX])^{\perp} (\omega\Omega[z_3^{-1}X])^{\bullet} (\omega\Omega[-z_3MX])^{\perp}$$

$$= \Omega[-z_1Mz_2^{-1}]\, (\omega\Omega[z_1^{-1}X])^{\bullet} (\omega\Omega[z_2^{-1}X])^{\bullet} (\omega\Omega[-z_1MX])^{\perp} (\omega\Omega[-z_2MX])^{\perp} (\omega\Omega[z_3^{-1}X])^{\bullet} (\omega\Omega[-z_3MX])^{\perp}$$

① create $\Omega[-z_2Mz_3^{-1}]$

② create $\Omega[-z_1Mz_3^{-1}]$

$$= \Omega[-z_1Mz_2^{-1}]\,\Omega[-z_1Mz_3^{-1}]\,\Omega[-z_2Mz_3^{-1}]\, \underbrace{(\omega\Omega[z_1^{-1}X])^{\bullet}(\omega\Omega[z_2^{-1}X])^{\bullet}(\omega\Omega[z_3^{-1}X])^{\bullet}}_{\omega\Omega[(z_1^{-1}+z_2^{-1}+z_3^{-1})X]^{\bullet}}\, \underbrace{(\omega\Omega[-z_1MX])^{\perp}(\omega\Omega[-z_2MX])^{\perp}(\omega\Omega[-z_3MX])^{\perp}}_{(\omega\Omega[-(z_1+z_2+z_3)MX])^{\perp}}$$

$$\therefore\ D(z_1)D(z_2)\cdots D(z_\ell)\cdot 1 = \Big(\prod_{1\le i<j\le \ell} \Omega[-\tfrac{z_i}{z_j}M]\Big)(\omega\Omega[(z_1^{-1}+z_2^{-1}+\cdots+z_\ell^{-1})X])^{\bullet}\, \underbrace{(\omega\Omega[-(z_1+\cdots+z_\ell)MX])^{\perp}\cdot 1}_{1}$$

$$= \prod_{1\le i<j\le \ell} \Omega[-\tfrac{z_i}{z_j}M]\,(\omega\Omega[(z_1^{-1}+z_2^{-1}+\cdots+z_\ell^{-1})X])$$

$$\therefore\ D_{a_1}D_{a_2}\cdots D_{a_\ell}\cdot 1 = \langle z_1^{-a_1} z_2^{-a_2} \cdots z_\ell^{-a_\ell} \rangle\,(D(z_1)D(z_2)\cdots D(z_\ell)\cdot 1)$$

Recall $\Omega[a_1+a_2+\cdots-b_1-b_2-\cdots] = \dfrac{\prod_i (1-b_i)}{\prod_i (1-a_i)}$

$$= \langle z_1^{-a_1} z_2^{-a_2} \cdots z_\ell^{-a_\ell} \rangle \prod_{1\le i<j\le \ell} \Omega[-\tfrac{z_i}{z_j}M]\,(\omega\Omega[(z_1^{-1}+z_2^{-1}+\cdots+z_\ell^{-1})X])$$

$$\Rightarrow \omega(D_{a_1}D_{a_2}\cdots D_{a_\ell}\cdot 1)(x_1,\ldots,x_\ell) = \langle z_1^{-a_1} z_2^{-a_2} \cdots z_\ell^{-a_\ell} \rangle \prod_{1\le i<j\le \ell} \Omega[-\tfrac{z_i}{z_j}M]\,\Omega[(z_1^{-1}+\cdots+z_\ell^{-1})X]$$

$$\Omega[-\tfrac{z_i}{z_j}M] = \Omega[-\tfrac{z_i}{z_j}(1-q)(1-t)] = \Omega[q\tfrac{z_i}{z_j}+t\tfrac{z_i}{z_j}-qt\tfrac{z_i}{z_j}-\tfrac{z_i}{z_j}] = \frac{(1-qt\frac{z_i}{z_j})(1-\frac{z_i}{z_j})}{(1-q\frac{z_i}{z_j})(1-t\frac{z_i}{z_j})}$$

$$= \underbrace{\langle z^0 \rangle}_{\text{(const. term of)}} \Big(z_1^{a_1} z_2^{a_2} \cdots z_\ell^{a_\ell} \prod_{1\le i<j\le \ell} \frac{(1-qt\frac{z_i}{z_j})(1-\frac{z_i}{z_j})}{(1-q\frac{z_i}{z_j})(1-t\frac{z_i}{z_j})}\,\Omega[(z_1^{-1}+\cdots+z_\ell^{-1})X]\Big)$$

$$= \langle z^0 \rangle \Big(z^{\vec{a}} \prod_{1\le i<j\le \ell} \frac{1-qt\frac{z_i}{z_j}}{(1-q\frac{z_i}{z_j})(1-t\frac{z_i}{z_j})}\,\Omega[\overline{Z}X] \prod_{i<j}(1-\tfrac{z_i}{z_j})\Big) \qquad \overline{Z} = \overline{z_1+z_2+\cdots+z_\ell}$$

Cauchy on $\Omega[\overline{Z}X] = \sum s_\lambda[\overline{Z}]\, s_\lambda(X)$

$$= \sum_\lambda s_\lambda(X)\cdot \langle z^0 \rangle\, z^{\vec{a}} \prod_{1\le i<j\le \ell} \frac{(1-qt\frac{z_i}{z_j})}{(1-q\frac{z_i}{z_j})(1-t\frac{z_i}{z_j})}\, s_\lambda[\overline{Z}] \cdot \prod_{i<j}(1-\tfrac{z_i}{z_j})$$

since there are ℓ variables and we only care about const. term, we can still apply:

$\therefore$ we can change z to x

(Lemma 2.3.1 (ii))

$$\langle \chi_\lambda \rangle \sigma(\phi(x)) = \langle x^0 \rangle\, \overline{\chi_\lambda}\, \phi(x) \prod_{1\le i<j\le \ell} (1-\tfrac{x_i}{x_j})$$

$s_\lambda[\overline{Z}]$, $\prod_{1\le i<j\le \ell} \dfrac{z^{\vec{a}}(1-qt\frac{z_i}{z_j})}{(1-q\frac{z_i}{z_j})(1-t\frac{z_i}{z_j})}$, $1-\frac{z_i}{z_j}$

even our $\phi(x)$ is not a Laurent polynomial

$$= \sum_{\lambda} s_\lambda(X) \cdot \langle \chi_\lambda \rangle \, \sigma\left(x^{\vec{a}} \cdot \prod_{1 \le i < j \le \ell} \frac{1 - qt\frac{x_i}{x_j}}{\left(1 - q\frac{x_i}{x_j}\right)\left(1 - t\frac{x_i}{x_j}\right)} \right)$$

$$= \sigma\left(x^{\vec{a}} \prod_{1 \le i < j \le \ell} \frac{1 - qt\frac{x_i}{x_j}}{\left(1 - q\frac{x_i}{x_j}\right)\left(1 - t\frac{x_i}{x_j}\right)} \right)_{\mathrm{pol}}$$

$$= H^{\ell}_{q,t}(x^{\vec{a}})_{\mathrm{pol}}$$

$\square$

* Since Z has only ℓ variables, all Schur functions s_λ in $\omega(\xi \cdot 1)(x_1, \ldots, x_\ell)$ has $\ell(\lambda) \le \ell$.
Hence $\xi \cdot 1$ is determined by $\omega(\xi \cdot 1)$.