· For a box $(i,j)$ (row $j$, column $i$) of a skew Young diagram, content $(c)$ $= i-j$.

· Fix $\varepsilon>0$ s.t $k\varepsilon<1$ where $k$ = # skew Young diagrams. Then adjusted content $(\tilde{c})$ = content + $r\varepsilon$, where the box is in the $r^{th}$ diagram/shape.

Reading order: any total ordering of the boxes on which adjusted content is increasing.

i.e. lexicographic · first by content, then by index of the diagrams. (boxes with the same content in the same skew diagram can be ordered arbitrarily).

* We write $\nu=(\nu^{(1)},\dots,\nu^{(k)})$ : a tuple of k skew Young diagrams.

Hence a, b are in different $\nu^{(i)}$.

- Let $a,b\in\nu$ be two boxes in $\nu$. a, b attack each other if $0<|\tilde{c}(a)-\tilde{c}(b)|<1$.

  If $a\in\nu^{(i)}$, $b\in\nu^{(j)}$ with a preceding b in reading order, then a, b attack if either:

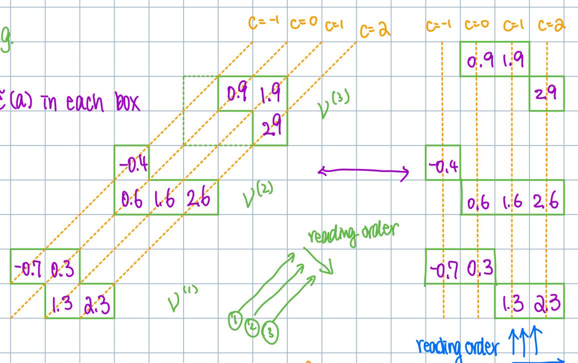
  ① $c(a)=c(b)$ and $i<j$

  ② $c(a)=c(b)-1$ and $i>j$

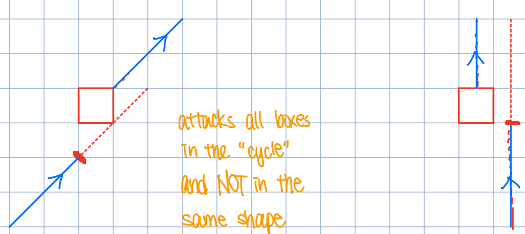
  We say that a, b form an attacking pair.

e.g

$\tilde{c}(a)$ in each box

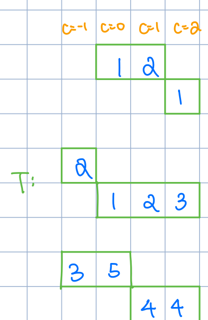



$k=3$, $\varepsilon=0.3$, $\nu=(32/1,\ 31,\ 33/21)$




Def: Attacking inversion is an attacking pair $(a,b)$ with a preceding b and entry in box a > entry in box b.

e.g




content of a

Inversions:

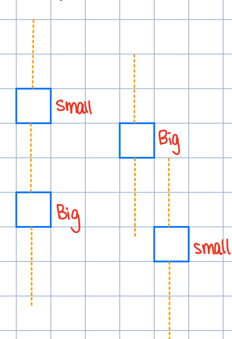
c=-1: $(\boxed{3},\boxed{2})$

c=0: $(\boxed{5},\boxed{1})$, $(\boxed{5},\boxed{1})$

c=1: $(\boxed{4},\boxed{2})$, $(\boxed{4},\boxed{2})$,

c=2: $(\boxed{4},\boxed{3})$, $(\boxed{4},\boxed{1})$, $(\boxed{3},\boxed{1})$

$\operatorname{inv}(T)=8$




corresponds to $q^8x_1^3x_2^3x_3^2x_4^2x_5$ in $G_{(32/1,\,31,\,33/21)}$

Def: The Combinatorial LLT polynomial indexed by a tuple of skew Young diagrams $\nu$ is the generating function

$$G_\nu(x;q)=\sum_{T\in SSYT(\nu)} q^{\operatorname{inv}(T)}x^T$$

Let $\mathcal{A} = \mathcal{A}_+ \sqcup \mathcal{A}_-$ be a 'signed' alphabet with $\mathcal{A}_+ = \{1, 2, 3, \ldots\}$ and $\mathcal{A}_- = \{\bar{1}, \bar{2}, \bar{3}, \ldots\}$, and an arbitrary total ordering on $\mathcal{A}$.

positive letters

negative letters

* In this paper, we choose the ordering $\cdots < \bar{2} < \bar{1} < 1 < 2 < 3 < \cdots$ (so $\bar{v}$ is like $-v$ $\forall v \in \mathbb{Z}^+$)

Def: A super tableau on a tuple of skew shapes $\nu = (\nu^{(1)}, \ldots, \nu^{(k)})$ is a map

$$T: \nu \longrightarrow \mathcal{A} \quad \text{(assigning each box in } \nu \text{ a letter in } \mathcal{A}\text{)}$$

s.t. ① rows and columns are weakly increasing (from left to right along rows, bottom to top along columns)

② positive letters in columns are strictly increasing (from left to right)

③ negative letters in rows are strictly increasing (from bottom to top)

∴ A SSYT is a super tableau with all entries positive.

Notation: $SSYT_{\pm}(\nu)$ := set of all super tableaux on $\nu$

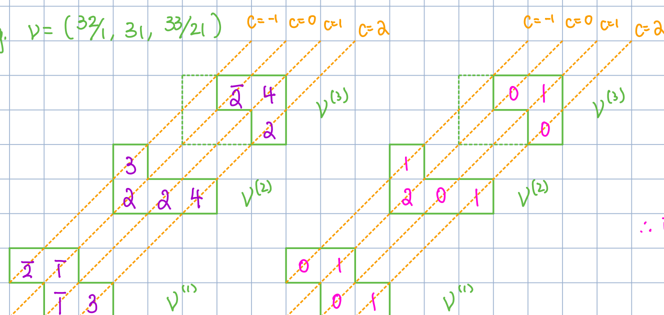
Def: An attacking inversion in a super tableau is an attacking pair $(a, b)$ with $a$ preceding $b$ in reading order, s.t. either

(i) $T(a) > T(b)$ in the ordering in $\mathcal{A}$, or

(ii) $T(a) = T(b) \in \mathcal{A}_-$ (equal and negative)

Hence $\operatorname{inv}(T)$ also defines for super tableaux $T$.

e.g. $\nu = (32/1,\ 31,\ 33/21)$

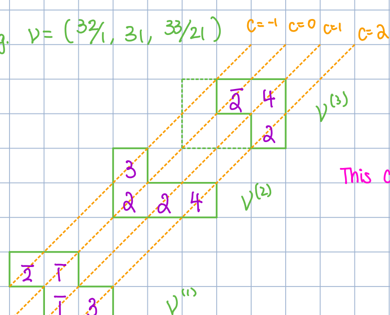



$\therefore \operatorname{inv}(T) = 7$

contribution to $\operatorname{inv}(T)$

Lemma 4.1.2: $\omega_Y \mathcal{G}_\nu[X+Y; q] = \sum_{T \in SSYT_{\pm}(\nu)} q^{\operatorname{inv}(T)} x^{T_+} y^{T_-}$ where $x^{T_+} y^{T_-} = \prod_{a \in \nu} \begin{cases} x_{T(a)} & \text{if } T(a) \in \mathcal{A}_+ \\ y_{T(a)} & \text{if } T(a) \in \mathcal{A}_- \end{cases}$.

e.g. $\nu = (32/1,\ 31,\ 33/21)$




This contributes $q^7 x_2^3 x_3^2 x_4^2 y_1^2 y_2^2$ to $\omega_Y \mathcal{G}_\nu[X+Y; q]$ with $\nu = (32/1,\ 31,\ 33/21)$.

Corollary 4.1.3: $\omega \mathcal{G}_\nu(X; q) = \sum_{T \in SSYT_-(\nu)} q^{\operatorname{inv}(T)} x^T$.

← T with only negative letters

Pf: Set $X = 0$ in Lemma 4.1.2. Then the only surviving terms on R.H.S. are those $T$ with no positive letters.

Hence $\omega_Y \mathcal{G}_\nu[0+Y; q] = \sum_{T \in SSYT_-(\nu)} q^{\operatorname{inv}(T)} y^{T_-} = \sum_{T \in SSYT_-(\nu)} q^{\operatorname{inv}(T)} \underbrace{\prod_{a \in \nu} y^{T(a)}}_{y^T} = \sum_{T \in SSYT_-(\nu)} q^{\operatorname{inv}(T)} y^T$.

Result follows by a change of variables $Y \mapsto X$. □

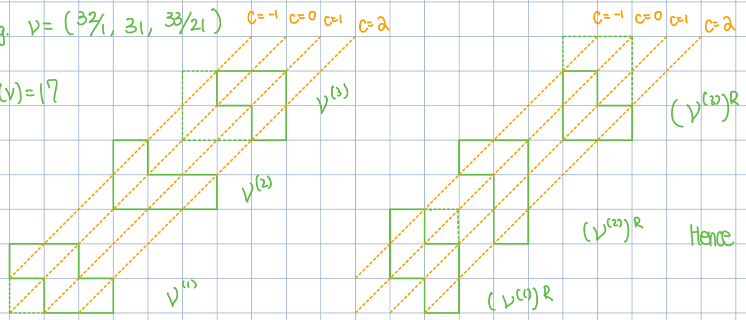
**Prop. 4.1.4:** Given a tuple of skew Young diagrams $\nu = (\nu^{(1)}, \dots, \nu^{(k)})$, let $\nu^R = ((\nu^{(1)})^R, \dots, (\nu^{(k)})^R)$, where $(\nu^{(i)})^R$ is the $180°$ rotation of $(\nu^{(i)})^*$ (transpose) positioned so that each box in $\nu^R$ has the same content as the corresponding box in $\nu$. Then

$$\omega \mathcal{G}_\nu(X;q) = q^{I(\nu)} \mathcal{G}_{\nu^R}(X;q^{-1})$$

where $I(\nu) = \#$ attacking pairs in $\nu$.

e.g. $\nu = (32/1,\ 31,\ 33/21)$

$I(\nu) = 17$

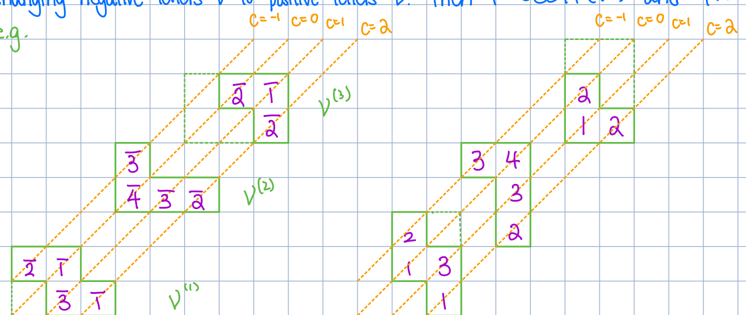



$\therefore\ \nu^R = (221/1,\ 222/11,\ 21)$

Hence $\omega \mathcal{G}_{(32/1,\,31,\,33/21)}(X;q) = q^{17} \mathcal{G}_{(221/1,\,222/11,\,21)}(X;q^{-1})$

**Proof:** Given a negative tableau $T$ on $\nu$, let $T^R$ be the tableau on $\nu^R$ obtained by reflecting the tableau along with $\nu$ and changing negative letters $\bar{v}$ to positive letters $v$. Then $T^R \in SSYT(\nu^R)$ and $T \mapsto T^R$ is weight preserving, and $I(\nu) = I(\nu^R)$.

e.g.




$T \in SSYT_-(\nu)$ $\qquad$ $T^R \in SSYT(\nu^R)$

$x^T = x^{\bar{T}} = x_1^3 x_2^4 x_3^3 x_4$ $\qquad$ $x^{T^R} = x_1^3 x_2^4 x_3^3 x_4$

An attacking pair in $\nu$ is an inversion in $T$ iff the corresponding attacking pair in $\nu^R$ is a non-inversion in $T^R$.
(Since contents are preserved, attacking pairs in $\nu$ is still an attacking pair in $\nu^R$.
Also $T(a) > T(b)$ for $T \in SSYT_-$ iff $\overline{T(a)} \le \overline{T(b)}$, equivalently, $T^R(a) < T^R(b)$ (in $\mathcal{A}_+$).
$\therefore$ inversion in $T \longleftrightarrow$ non-inversion in $T^R$)

$\therefore\ \operatorname{inv}(T) = I(\nu^R) - \operatorname{inv}(T^R) = I(\nu) - \operatorname{inv}(T^R)$.

By Corollary 4.1.3., $\omega \mathcal{G}_\nu(X;q) = \sum_{T \in SSYT_-(\nu)} q^{\operatorname{inv}(T)} x^T = \sum_{T^R \in SSYT(\nu^R)} q^{I(\nu) - \operatorname{inv}(T^R)} x^{T^R} = q^{I(\nu)} \mathcal{G}_{\nu^R}(X;q^{-1})$. $\square$

**Lemma 4.1.6:** $\mathcal{G}_\nu(X;q)$ is a linear combination of Schur functions $s_\lambda(X)$ s.t. $\ell(\lambda) \le$ Total no. of rows in $\nu$.

Proof: Let $r$ = total no. of rows in $\nu$. Then it is equivalent to show that $\omega \mathcal{G}_\nu(X;q)$ is a linear combination of $m_\lambda(X)$ s.t. $\lambda_1 \le r$. By Prop. 4.1.4, $\omega \mathcal{G}_\nu(X;q)$ has a monomial term $q^{I(\nu) - \operatorname{inv}(T)} x^T$ for each $T \in SSYT(\nu^R)$.
Since a letter can appear at most once in each column of $\nu^R$, the exponents of $x^T$ is bounded above by $\#$ columns in $\nu^R$ which is also the number of rows in $\nu$, i.e. $r$. $\square$