For GL_ℓ : (see §2 for details)

- ground field : $\Bbbk$ (any field containing $\mathbb{Q}(q)$)
- weight lattice : $X = \mathbb{Z}^\ell$
- group algebra : $\Bbbk X \cong \Bbbk[x_1^{\pm1}, x_2^{\pm1}, \dots, x_\ell^{\pm1}]$ $\qquad (a_1,\dots,a_\ell) \longleftrightarrow x_1^{a_1} x_2^{a_2} \cdots x_\ell^{a_\ell}$
- Weyl group : $W = S_\ell$
- roots : $\alpha_{ij} = \varepsilon_i - \varepsilon_j$ for $1 \le i \ne j \le \ell$
- positive roots : $\alpha_{ij} = \varepsilon_i - \varepsilon_j$ for $1 \le i < j \le \ell$ $\qquad R_+(GL_\ell) := \{\alpha_{ij} : 1 \le i < j \le \ell\}$ = set of all positive roots
- simple roots : $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \le i \le \ell - 1$

Def: The Demazure - Lusztig operators is defined as

$$T_i = q s_i + (1-q)\frac{1}{1-\frac{x_{i+1}}{x_i}}(s_i - 1) \qquad 1 \le i \le \ell - 1$$

They generate an action of the Hecke algebra $\mathcal{H}(S_\ell)$ on $\Bbbk[x_1^{\pm1}, \dots, x_\ell^{\pm1}]$.

Note: $(T_i - q)(T_i + 1) = 0$

Proof: $T_i - q = \left[q + (1-q)\frac{1}{1-\frac{x_{i+1}}{x_i}}\right](s_i - 1) = \frac{1 - q\frac{x_{i+1}}{x_i}}{1 - \frac{x_{i+1}}{x_i}}(s_i - 1)$

$$T_i + 1 = q s_i + \frac{s_i \cancel{-1} - q s_i + q \cancel{+1} - \frac{x_{i+1}}{x_i}}{1 - \frac{x_{i+1}}{x_i}} = q s_i + \frac{(1-q)s_i + q - \frac{x_{i+1}}{x_i}}{1 - \frac{x_{i+1}}{x_i}}$$

$$(T_i - q)(T_i + 1) = \left(\frac{1 - q\frac{x_{i+1}}{x_i}}{1 - \frac{x_{i+1}}{x_i}}(s_i - 1)\right)\left(q s_i + \frac{(1-q)s_i + q - \frac{x_{i+1}}{x_i}}{1 - \frac{x_{i+1}}{x_i}}\right)$$

$$= \frac{1 - q\frac{x_{i+1}}{x_i}}{1 - \frac{x_{i+1}}{x_i}}\left(q(1 - s_i) + \frac{1 - q + q s_i - \frac{x_i}{x_{i+1}} s_i}{1 - \frac{x_i}{x_{i+1}}} - \frac{(1-q)s_i + q - \frac{x_{i+1}}{x_i}}{1 - \frac{x_{i+1}}{x_i}}\right)$$

$$= \frac{1 - q\frac{x_{i+1}}{x_i}}{1 - \frac{x_{i+1}}{x_i}}\left(q - q s_i + \frac{\cancel{x_{i+1}} - q x_{i+1} + x_{i+1} \cdot q s_i \cancel{- x_i s_i}}{x_{i+1} - x_i} + \frac{x_i(\cancel{s_i} - q s_i) + q x_i \cancel{- x_{i+1}}}{x_{i+1} - x_i}\right)$$

$$= \frac{1 - q\frac{x_{i+1}}{x_i}}{1 - \frac{x_{i+1}}{x_i}}\left(q - q s_i + \underbrace{\frac{q(x_i - x_{i+1}) + (x_{i+1} - x_i) q s_i}{x_{i+1} - x_i}}_{-q + q s_i}\right) = 0$$

□

Alternative:

$T_i = q\pi_i - s_i\theta_i$ $\qquad$ (π_i : key operator, $\theta_i = \pi_i - 1$ = atom operator)

$$(T_i - q)(T_i + 1) = (q(\pi_i - 1) - s_i\theta_i)(q\pi_i - s_i\theta_i + 1)$$
$$= (q - s_i)\theta_i(q\pi_i - s_i\theta_i + 1)$$
$$= (q - s_i)(q\underbrace{\theta_i\pi_i}_{0} - \underbrace{\underbrace{\theta_i s_i}_{-\theta_i}\theta_i}_{-\theta_i^2 = \theta_i} + \theta_i) = (q - s_i)(-\theta_i + \theta_i)$$
$$= 0$$

For $w \in S_\ell$ with reduced expression $w = s_{i_1} \cdots s_{i_m}$, define $T_w := T_{i_1} T_{i_2} \cdots T_{i_m}$ (well-defined because T_i's satisfy the Braid relations $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ and $T_i T_j = T_j T_i$ $\forall |i-j|>1$)

* $\{T_w : w \in S_\ell\}$ form a $\Bbbk$-basis of the Hecke algebra.

Set $R_+ = R_+(GL_\ell)$ (positive roots)

$Q = \mathbb{Z}\{\alpha_{ij} : 1 \le i \ne j \le \ell\} = \left\{\sum_{i=1}^{\ell} a_i\varepsilon_i : a_i \in \mathbb{Z},\ \sum_{i=1}^{\ell} a_i = 0\right\}$. (root lattice)

$Q_+ = \mathbb{N}R_+ = \mathrm{span}_{\mathbb{N}}\{\alpha_{ij} : 1 \le i < j \le \ell\}$ (cone in the root lattice Q)

Recall : $(\lambda_1, \dots, \lambda_\ell) \in \mathbb{Z}^\ell$: dominant if $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_\ell$

: regular if $\lambda_i \ne \lambda_j$ $\forall i \ne j$

has trivial stabilizer in S_ℓ

For dominant weights, we define

$\mu \ge \lambda$ if $\mu - \lambda \in Q_+$ (if λ, μ are partitions, this is the same as dominance order on partitions)

e.g. $(4,4,3,2) \ge (4,3,2,2,2)$ in dominance order

$(4,4,3,2) - (4,3,2,2,2) = (0,1,1,0,-2) = \alpha_{25} + \alpha_{35} \in Q_+$

Notation: For $\lambda \in \mathbb{Z}^\ell$, denote λ_+ = dominant weight in the orbit $S_\ell \cdot \lambda$ (i.e. permute entries s.t. they are weakly decreasing).

Let $\mathrm{conv}(S_\ell \cdot \lambda)$ be the convex hull of the orbit $S_\ell \cdot \lambda$ in the coset $\lambda + Q$ of the root lattice, i.e. the set of weights that occur with nonzero multiplicity in the irreducible character χ_{λ_+}

e.g. $\ell=2$, $Q=\{(a,-a) : a\in\mathbb{Z}\}$. Take $\lambda=(10,-2)$ $\therefore \lambda=\lambda_+$ (same result if we choose $\lambda=(-2,10)$)

Then $S_2\cdot\lambda = \{(10,-2),(-2,10)\}$

$\lambda+Q = \{(10+a,-2-a) : a\in\mathbb{Z}\}$.

Hence orbit $S_2\cdot\lambda$ has endpoints with $a=0$ and $a=-12$

$\therefore \text{conv}(S_2\cdot\lambda) = \{(10+a,-2-a) : -12\le a\le 0\}$

$= \{(-2,10),(-1,9),(0,8),(1,7),(2,6),(3,5),(4,4),(5,3),(6,2),(7,1),(8,0),(9,-1),(10,-2)\}$

Note that $\chi_{\lambda_+} = x_1^{10}x_2^{-2}+x_1^{9}x_2^{-1}+x_1^{8}x_2^{0}+x_1^{7}x_2+x_1^{6}x_2^{2}+x_1^{5}x_2^{3}+x_1^{4}x_2^{4}+x_1^{3}x_2^{5}+x_1^{2}x_2^{6}+x_1x_2^{7}+x_1^{0}x_2^{8}+x_1^{-1}x_2^{9}+x_1^{-2}x_2^{10}$

$= \sum_{(a,b)\in\text{conv}(S_2\cdot\lambda)} x_1^a x_2^b$ (Hence these terms are the non-vanishing terms in χ_{λ_+})

Note that $\text{conv}(S_\ell\cdot\lambda)\subseteq\text{conv}(S_\ell\cdot\mu)$ iff $\lambda_+\le\mu_+$ (b/c $S_\ell\cdot\lambda_+\subseteq\text{conv}(S_\ell\cdot\mu)$ iff $\lambda_+\le\mu_+$)

e.g. $\mu=(12,-4) > \lambda=(10,-2)$ b/c $\mu-\lambda=(2,-2)=2\alpha_{12}\in Q_+$ ($\mu_+=\mu$ in this case)

$S_2\cdot\mu = \{(12,-4),(-4,12)\}$

$\mu+Q = \{(12+a,-4-a) : a\in\mathbb{Z}\}$

Hence orbit $S_2\cdot\mu$ has endpoints with $a=0$ and $a=-16$

$\text{conv}(S_2\cdot\mu) = \{(12+a,-4-a) : -16\le a\le 0\}$

$= \{(-4,12),(-3,11),(-2,10),(-1,9),(0,8),(1,7),(2,6),(3,5),(4,4),(5,3),(6,2),(7,1),(8,0),(9,-1),(10,-2),(11,-3),(12,-4)\}$

$= \{(-4,12),(-3,11),(11,-3),(12,-4)\}\sqcup\text{conv}(S_2\cdot\lambda)$

$\therefore \text{conv}(S_2\cdot\lambda)\subseteq\text{conv}(S_2\cdot\mu)$

Each orbit $S_\ell\cdot\lambda_+$ has a partial ordering induced by the Bruhat ordering on S_ℓ.

This ordering is the transitive closure of the relation $s_i\lambda>\lambda$ if $\langle\alpha_i^\vee,\lambda\rangle>0$.

e.g. $\lambda=(2,4,1,5,2)$ $\langle\alpha_1^\vee,\lambda\rangle=2-4=-2<0$, $\langle\alpha_2^\vee,\lambda\rangle=4-1=3>0$, $\langle\alpha_3^\vee,\lambda\rangle=1-5=-4<0$, $\langle\alpha_4^\vee,\lambda\rangle=5-2=3>0$

Hence $s_2\cdot\lambda>\lambda$ and $s_4\cdot\lambda>\lambda$

ie $(2,1,4,5,2)>(2,4,1,5,2)$ and $(2,4,1,2,5)>(2,4,1,5,2)$

Similarly, $(1,2,4,5,2)>(2,1,4,5,2)>(2,4,1,5,2)$

We extend this to all of $\mathbb{Z}^\ell$ (instead of just within $S_\ell\cdot\lambda$) by defining

$\lambda\le\mu$ if $\lambda_+<\mu_+$ or ($\lambda_+=\mu_+$ and $\lambda\le\mu$ in the Bruhat order on $\underset{\text{b/c }\lambda_+=\mu_+}{S_\ell\cdot\lambda_+}$)

e.g. • We knew $(2,4,1,5,2)\le(2,4,1,2,5)$ b/c $\langle\alpha_4^\vee,(2,4,1,5,2)\rangle=5-2=3>0 \Rightarrow (2,4,1,2,5)\ge(2,4,1,5,2)$

• Take $\lambda=(2,4,1,5,2)$, $\mu=(3,5,-1,5,2)$

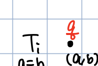
Then $\lambda_+=(5,4,2,2,1)$, $\mu_+=(5,5,3,2,-1)$

$\mu_+-\lambda_+=(0,1,1,0,-2)=\alpha_{25}+\alpha_{35}\in Q_+ \Rightarrow \mu_+>\lambda_+$

Hence $(2,4,1,5,2)\le(3,5,-1,5,2)$

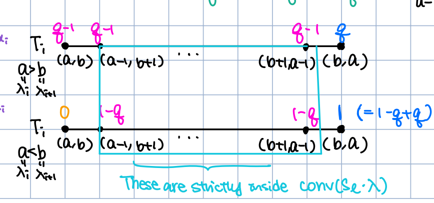
Suppose $\langle\alpha_i^\vee,\lambda\rangle\ge 0$.

Case I) $\langle\alpha_i^\vee,\lambda\rangle=0$, i.e. $s_i\lambda=\lambda$. Then $T_ix^\lambda=qx^\lambda$ (b/c $(s_i-1)\lambda=0 \Rightarrow T_ix^\lambda=qs_i\cdot x^\lambda=qx^{s_i\lambda}=qx^\lambda$)



Case II) $\langle\alpha_i^\vee,\lambda\rangle>0$. Then $T_ix^\lambda=qx^{s_i\lambda}+(q-1)\sum_{k=0}^{\lambda_i-\lambda_{i+1}-1}x^{\lambda-k\alpha_i}$

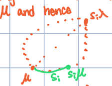
$T_ix^{s_i\lambda}=x^\lambda+(1-q)\sum_{k=1}^{\lambda_i-\lambda_{i+1}-1}x^{\lambda-k\alpha_i}$



Consider μ s.t. $\mu \leq s_i\lambda$

- If $\mu_+ < (s_i\lambda)_+$, since $(s_i\mu)_+ = \mu_+$, we have $(s_i\mu)_+ < (s_i\lambda)_+$ and hence $s_i\mu < s_i\lambda$.

- If $\mu_+ = (s_i\lambda)_+$ (then $(s_i\mu)_+ = \mu_+ = (s_i\lambda)_+$) and $\mu \leq s_i\lambda$ in Bruhat order on $S_\ell \cdot \mu_+$, and if $\langle \alpha_i^\vee, \mu \rangle \leq 0$, then $\langle \alpha_i^\vee, s_i\mu \rangle \geq 0$ and hence $s_i\mu \leq \mu \leq s_i\lambda$.
  If $\langle \alpha_i^\vee, \mu \rangle > 0$, then as $\langle \alpha_i^\vee, \lambda \rangle > 0$, we know $\langle \alpha_i^\vee, s_i\lambda \rangle < 0$. Hence $\langle \alpha_i^\vee, \mu \rangle > 0$ and $\mu \leq s_i\lambda$ with $\langle \alpha_i^\vee, \lambda \rangle < 0$ means $s_i\mu \leq s_i\lambda$.
  (b/c there is a path from μ to $s_i\lambda$ and s_i must be involved in the path b/c $\mu_i > \mu_{i+1}$ but $(s_i\lambda)_i < (s_i\lambda)_{i+1}$. Then there is a path with the same length that starts with s_i from μ to $s_i\mu$ and hence



  i.e. $s_i\mu \leq s_i\lambda$.)

Hence $s_i\mu \leq s_i\lambda$.

As a result, the set $\{x^\mu : \mu \leq s_i\lambda\}$ is s_i-invariant.

Given any root ν, $(\nu + \mathbb{Z}\alpha_i) \cap \{\mu : \mu \leq s_i\lambda\}$ is convex. i.e. if $\nu + k_1\alpha_i, \nu + k_2\alpha_i \leq s_i\lambda$ for some $k_1 \leq k_2 \in \mathbb{Z}$, then $\nu + k\alpha_i \leq s_i\lambda$ $\forall\, k_1 \leq k \leq k_2$.
b/c $(\nu + k\alpha_i)_+ \leq (\nu + k_2\alpha_i)_+ \leq s_i\lambda$

Hence $\mathbb{k} \cdot \{x^\mu : \mu \leq s_i\lambda\}$ is closed under T_i.

$\therefore$ For $\langle \alpha_i^\vee, \lambda \rangle \geq 0$ and $c_\mu \in \mathbb{k}$ $\forall \mu < \lambda$, we have:

$$T_i\Big(x^\lambda + \sum_{\mu < \lambda} c_\mu x^\mu\Big) = q x^{s_i\lambda} + \sum_{\mu < s_i\lambda} d_\mu x^\mu \quad \text{for some } d_\mu \in \mathbb{k}$$

Then $\mu < \lambda \leq s_i\lambda$