Def: The non-symmetric Hall-Littlewood polynomial is defined as

$$E_\lambda(x;q) = E_\lambda(x_1,\dots,x_\ell;q) := q^{-\ell(\omega)} T_\omega x^{\lambda_+}$$

where $\lambda \in \mathbb{Z}^\ell$ (GL_ℓ weight) and $\omega \in S_\ell$ s.t. $\lambda = \omega(\lambda_+)$ (ω is not unique if λ has equal parts, but E_λ is independent of the choice of ω b/c $T_i x^\lambda = q x^\lambda$ for $s_i\lambda = \lambda$ and the extra q factor is canceled by $q^{-\ell(\omega)}$ if ω has a longer length)

Remark: $E_\lambda(x;q)$ is monic and triangular:

Also, $c_\mu \in \mathbb{Q}(q)$

$$E_\lambda(x;q) = x^\lambda + \sum_{\mu<\lambda} c_\mu x^\mu \left(\text{Recall } T_i(x^\lambda + \sum_{\mu<\lambda} \tilde{c}_\mu x^\mu) = q x^{s_i\lambda} + \sum_{\mu \le s_i\lambda} d_\mu x^\mu \ \forall c_\mu \in \mathbb{K}. \text{ Thus, } E_\lambda(x;q) = q^{-\ell(\omega)} T_\omega x^{\lambda_+} = q^{-\ell(\omega)}\Big(q^{\ell(\omega)} x^{\omega(\lambda_+)} + \sum_{\mu<\omega(\lambda_+)} d'_\mu x^\mu\Big) = x^\lambda + \sum_{\mu<\lambda} c_\mu x^\mu\right)$$

e.g. $\lambda = (4,9,7)$ $\ell = 3$ $\Rightarrow \lambda_+ = (9,7,4)$, $\omega = s_1 s_2$ check: $s_2\cdot\lambda_+ = (9,4,7)$ $s_1 s_2 \cdot \lambda_+ = s_1\cdot(9,4,7) = (4,9,7) = \lambda$

$\therefore$ $E_{(4,9,7)}(x_1,x_2,x_3;q) = q^{-\ell(\omega)} T_\omega x^{974}$

$$= q^{-2} T_1 T_2 x^{974}$$

$$= q^{-2} T_1 \left((q-1)x^{974} + (q-1)x^{965} + (q-1)x^{956} + q x^{947}\right)$$

$$= q^{-2}(q-1)\left[(q-1)x^{974} + (q-1)x^{884} + q x^{794}\right] + q^{-2}(q-1)\left[(q-1)x^{965} + (q-1)x^{875} + (q-1)x^{785} + q x^{695}\right] + q^{-2}(q-1)\left[(q-1)x^{956} + (q-1)x^{866} + (q-1)x^{776} + (q-1)x^{686} + q x^{596}\right]$$
$$+ q^{-1}\left[(q-1)x^{947} + (q-1)x^{857} + (q-1)x^{767} + (q-1)x^{677} + (q-1)x^{587} + q x^{497}\right]$$

$$= q^{-2}(q-1)^2(x^{974} + x^{884} + x^{965} + x^{875} + x^{785} + x^{956} + x^{866} + x^{776} + x^{686}) + q^{-1}(q-1)(x^{794} + x^{695} + x^{596} + x^{947} + x^{857} + x^{767} + x^{677} + x^{587}) + x^{497}$$

all μ in x^μ satisfy $\mu < 497$

$$= \underset{\text{monic}}{x^{497}} + \underbrace{(1-q^{-1})}_{\in \mathbb{Q}[q^{-1}]}(x^{794} + x^{695} + x^{596} + x^{947} + x^{857} + x^{767} + x^{677} + x^{587}) + (1-q^{-1})^2(x^{974} + x^{884} + x^{965} + x^{875} + x^{785} + x^{956} + x^{866} + x^{776} + x^{686})$$

(F-version)

$$F_\lambda(x;q) = F_\lambda(x_1,\dots,x_\ell;q) := T_\omega x^{\lambda_-}$$

longest permutation in S_ℓ

where $\lambda \in \mathbb{Z}^\ell$ (GL_ℓ weight), λ_-: weakly decreasing parts of λ (i.e. $\lambda_- = \omega_0(\lambda_+)$) and $\omega \in S_\ell$ s.t. $\lambda = \omega(\lambda_-)$

e.g. $\lambda = (4,9,7)$ $\ell = 3$ $\Rightarrow \lambda_- = (4,7,9)$, $\omega = s_2$

$\therefore$ $F_{497}(x;q) = T_2(x^{479}) = (1-q)x^{488} + x^{497}$

coeff $\in \mathbb{Q}[q]$

Remark: $q \to \infty$: E_λ are Demazure characters / key polynomials

$q \to 0$: F_λ are Demazure atoms (may need to reverse the variables)

Recall $Inv(\sigma) = \{(i,j) : i<j \text{ and } \sigma(i) > \sigma(j)\}$ for any $\sigma \in S_\ell$. We extend this to $\mathbb{R}^\ell$: $Inv(\mu) := \{(i,j) : i<j,\ \mu_i > \mu_j\}$, $\mu \in \mathbb{R}^\ell$.

Take ρ s.t. $\langle \alpha_i^\vee, \rho\rangle = 1$ (say $\rho = (\ell-1,\dots,1,0)$) and take $\varepsilon > 0$ (small), then

$$Inv(\lambda + \varepsilon\rho) = \{(i,j) : i<j \text{ and } \lambda_i \geq \lambda_j\}.$$

b/c $(\lambda+\varepsilon\rho)_i > (\lambda+\varepsilon\rho)_j \Leftrightarrow \lambda_i - \lambda_j > \varepsilon(\rho_j - \rho_i) = \varepsilon(i-j)$ and $\varepsilon > 0$ small $\Rightarrow \lambda_i - \lambda_j \geq 0$)

Def: Let $\sigma \in S_\ell$. The twisted non-symmetric Hall-Littlewood polynomials is defined as

b/c T_i acts on triangular expressions gives triangular expressions

- $E_\lambda^\sigma(x;q) = q^{|Inv(\sigma^{-1}) \cap Inv(\lambda+\varepsilon\rho)|} T_{\sigma^{-1}}^{-1} E_{\sigma^{-1}\lambda}(x;q)$ (since $T_i^{-1} = q^{-1}T_i + (q^{-1}-1)$ and E_μ is triangular $\forall \mu$, E_λ^σ is also triangular.)

- $F_\lambda^\sigma(x;q) = \overline{E_{-\lambda}^{\sigma w_0}(x;q)} = E_{-\lambda}^{\sigma(w_0)}(x_1^{-1}, x_2^{-1}, \dots, x_\ell^{-1}; q^{-1})$

where $w_0(i) = \ell+1-i$ $\forall\ 1 \le i \le \ell$. $\left(\sigma = \begin{pmatrix} \cdots a \cdots b \cdots \\ \cdots i \cdots i{+}1 \cdots \end{pmatrix}\ (a<b),\ s_i\sigma = \begin{pmatrix} \cdots a \cdots b \cdots \\ \cdots i{+}1 \cdots i \cdots \end{pmatrix},\ \sigma^{-1} = \begin{pmatrix} \cdots i\ i{+}1 \cdots \\ \cdots a\ b \cdots \end{pmatrix},\ \sigma^{-1}s_i = \begin{pmatrix} \cdots i\ i{+}1 \cdots \\ \cdots b\ a \cdots \end{pmatrix}\ (b>a)\right)$

Suppose $s_i\sigma > \sigma$, then

$(s_i \cdot Inv(\sigma^{-1})) \sqcup \{(i,i+1)\}$

$$E_{s_i\lambda}^{s_i\sigma}(x;q) = q^{|Inv(\sigma^{-1}s_i) \cap Inv(s_i\lambda+\varepsilon\rho)|} \underset{\substack{\| \\ (T_{\sigma^{-1}}T_i)^{-1}}}{T_{(s_i\sigma)^{-1}}^{-1}} E_{\underbrace{(s_i\sigma)^{-1}\cdot(s_i\lambda)}_{\sigma^{-1}\cdot\lambda}}(x;q)$$

$(i,j) \in Inv(\sigma^{-1}) \Leftrightarrow (i+1,j) \in Inv(\sigma^{-1}s_i)$
$(i+1,j) \in Inv(\sigma^{-1}) \Leftrightarrow (i,j) \in Inv(\sigma^{-1}s_i)$
$(j,i) \in Inv(\sigma^{-1}) \Leftrightarrow (j,i+1) \in Inv(\sigma^{-1}s_i)$
$(j,i+1) \in Inv(\sigma^{-1}) \Leftrightarrow (j,i) \in Inv(\sigma^{-1}s_i)$
$\Rightarrow Inv(\sigma^{-1}s_i) = \underline{\underline{s_i Inv(\sigma)}} \sqcup (i,i+1)$

$$|Inv(\sigma^{-1}s_i) \cap Inv(s_i\lambda+\varepsilon\rho)| = (s_i Inv(\sigma^{-1}) \sqcup \{(i,i+1)\}) \cap Inv(s_i\lambda+\varepsilon\rho) = (s_i Inv(\sigma^{-1}) \cap Inv(s_i\lambda+\varepsilon\rho)) \sqcup (\{(i,i+1)\} \cap Inv(s_i\lambda+\varepsilon\rho))$$
$$= |Inv(\sigma^{-1}) \cap Inv(\lambda+\varepsilon\rho)| + \mathbb{1}((s_i\lambda)_i \geq (s_i\lambda)_{i+1})$$
$$= |Inv(\sigma^{-1}) \cap Inv(\lambda+\varepsilon\rho)| + \mathbb{1}(\lambda_{i+1} \geq \lambda_i)$$

$\therefore\ E^{s_i\sigma}_{s_i\lambda}(x;q) = q^{|\mathrm{Inv}(\sigma^{-1})\cap \mathrm{Inv}(\lambda+\epsilon\rho)| + \mathbb{1}(\lambda_{i+1}\geq\lambda_i)} T_i^{-1} T_{\sigma^{-1}} E_{\sigma^{-1}\cdot\lambda}(x;q)$

$$= q^{\mathbb{1}(\lambda_{i+1}\geq\lambda_i)} T_i^{-1} E^{\sigma}_{\lambda}(x;q)$$

Hence $E^{\sigma}_{\lambda}(x;q) = q^{-\mathbb{1}(\lambda_{i+1}\geq\lambda_i)} T_i E^{s_i\sigma}_{s_i\lambda}(x;q)$ if $s_i\sigma>\sigma$

Similarly, $E^{\sigma}_{\lambda}(x;q) = q^{\mathbb{1}(\lambda_i\geq\lambda_{i+1})} T_i^{-1} E^{s_i\sigma}_{s_i\lambda}(x;q)$

Treat $s_i\sigma$ as σ
$s_i\lambda$ as λ

Then $\lambda_{i+1}\geq\lambda_i$ becomes $\lambda_{i+1}\leq\lambda_i$
b/c it means $(s_i\lambda)_{i+1}\geq(s_i\lambda)_i$
i.e. $\lambda_i\geq\lambda_{i+1}$

Thus we have the recurrence:

$$E^{\sigma}_{\lambda} = \begin{cases} q^{-\mathbb{1}(\lambda_{i+1}\geq\lambda_i)} T_i E^{s_i\sigma}_{s_i\lambda}(x;q) & \text{if } s_i\sigma>\sigma \\ q^{\mathbb{1}(\lambda_i\geq\lambda_{i+1})} T_i^{-1} E^{s_i\sigma}_{s_i\lambda}(x;q) & \text{if } s_i\sigma<\sigma \end{cases}$$

Note: (1) $E^{\sigma}_{\lambda_+} = x^{\lambda_+}\quad \forall\sigma\in S_\ell$

(2) If $\sigma=id$, then $E^{\sigma}_{\lambda}=E_\lambda$, $F^{\sigma}_{\lambda}=F_\lambda$.

Check: • $q^{|\mathrm{Inv}(\sigma^{-1})\cap\mathrm{Inv}(\lambda_+ +\epsilon\rho)|} = q^{|\mathrm{Inv}(\sigma^{-1})|} = q^{\ell(\sigma^{-1})}$ b/c $\mathrm{Inv}(\lambda_+ +\epsilon\rho) = \{(i,j) : 1\leq i<j\leq\ell\}$

• $E_{\sigma^{-1}\lambda_+}(x;q) = q^{-\ell(w)} T_w x^{(\sigma^{-1}\lambda)_+}$ where $w(\underbrace{(\sigma^{-1}\lambda)_+}_{\lambda_+}) = \sigma^{-1}\lambda_+ \Rightarrow w=\sigma^{-1}$

$$= q^{-\ell(\sigma^{-1})} T_{\sigma^{-1}} x^{\lambda_+}$$

$\therefore\ E^{\sigma}_{\lambda_+} = q^{\ell(\sigma^{-1})-\ell(\sigma^{-1})} T_{\sigma^{-1}}^{-1} T_{\sigma^{-1}} x^{\lambda_+} = x^{\lambda_+}$ which proves (1).

Since $\sigma^{-1}=id$ for $\sigma=id$, $\mathrm{Inv}(\sigma^{-1})=\phi$. Hence $E^{id}_{\lambda} = q^0 T_{id}^{-1} E_{id\cdot\lambda} = E_\lambda$

$F^{id}_{\lambda} = E^{w_0}_{-\lambda}(x_1^{-1},x_2^{-1},\dots,x_\ell^{-1};q^{-1})$ $\because \mathrm{Inv}(w_0^{-1}) = \mathrm{Inv}(w_0) = \{(i,j) : 1\leq i<j\leq\ell\}$

$\therefore \mathrm{Inv}(w_0^{-1})\cap\mathrm{Inv}(-\lambda+\epsilon\rho) = \mathrm{Inv}(-\lambda+\epsilon\rho) = \{(i,j) : i<j,\ -\lambda_i>-\lambda_j\} = \{(i,j) : i<j,\ \lambda_i<\lambda_j\}$

$\therefore E^{w_0}_{-\lambda}(x_1^{-1},\dots,x_\ell^{-1};q^{-1}) = (q^{-1})^{|\mathrm{Inv}(-\lambda+\epsilon\rho)|} T_{w_0}^{-1} E_{w_0\cdot(-\lambda)}(x^{-1};q^{-1})$

$$= q^{-|\mathrm{Inv}(-\lambda+\epsilon\rho)|} T_{w_0}^{-1}\left(q^{-(-\ell(w))} T_w x^{-\overset{-\lambda_-}{\overbrace{(-\lambda)_+}}}\right)$$

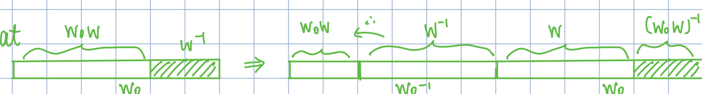
where $w_0\cdot(-\lambda) = w((-\lambda)_+) = -w(\lambda_-) \Rightarrow w_0(\lambda) = w(\lambda_-) \Rightarrow \boxed{\lambda = w_0w(\lambda_-)} \Rightarrow \ell(w)+\ell(w_0w) = \binom{\ell}{2}$

$\mathrm{Inv}(-\lambda+\epsilon\rho) = \{(i,j) : i<j,\ (-\lambda)_i\geq(-\lambda)_j\} = \{(i,j) : i<j,\ \lambda_i\leq\lambda_j\}$ $\therefore w_0w$ should not change position (i,j) $\Rightarrow |\mathrm{Inv}(-\lambda+\epsilon\rho)| + \ell(w_0w) = \binom{\ell}{2}$

$\therefore \ell(w) = |\mathrm{Inv}(-\lambda+\epsilon\rho)|$

$\therefore\ E^{w_0}_{-\lambda}(x_1^{-1},\dots,x_\ell^{-1};q^{-1}) = T_{w_0}^{-1} T_w x^{\lambda_-}$

Note that



i.e. $T_{w_0}^{-1}T_w = T_{w_0w}$

$\therefore\ F^{id}_{\lambda} = E^{w_0}_{-\lambda}(x_1^{-1},\dots,x_\ell^{-1};q) = T_{w_0w}x^{\lambda_-} = F_\lambda$

By (1) and recurrence, we can determine E_λ^σ $\forall\sigma$ and λ.

Note that $E^\sigma_{\lambda_+} = x^{\lambda_+}$ which is monic $\forall \sigma\in S_\ell$. (i.e. E^σ_λ is monic if $|\mathrm{Inv}(\lambda)|=0$)

We now show by induction on $|\mathrm{Inv}(\lambda)|$ that E^σ_λ is monic $\forall\lambda\in\mathbb{Z}^\ell, \sigma\in S_\ell$.

The statement is true for λ s.t. $|\mathrm{Inv}(\lambda)|=0$.

Assume the statement is true for λ s.t. $|\mathrm{Inv}(\lambda)|=k$ for some non-negative integer k.

Consider λ s.t. $|\mathrm{Inv}(\lambda)|=k+1$. ($\therefore \mathrm{Inv}(\lambda)\neq\phi$)

Take i s.t. $\lambda_{i+1}>\lambda_i$ (equivalently, $(s_i\lambda)_i > (s_i\lambda)_{i+1}$). (i.e. $i\in \mathrm{Inv}(\lambda)$)

If $s_i\sigma>\sigma$, then $E^\sigma_\lambda = q^{-1}T_i E^{s_i\sigma}_{s_i\lambda}$.

If $s_i\sigma<\sigma$, then $E^\sigma_\lambda = q^0 T_i^{-1} E^{s_i\sigma}_{s_i\lambda} = (q^{-1}T_i + (q^{-1}-1))E^{s_i\sigma}_{s_i\lambda} = q^{-1}T_iE^{s_i\sigma}_{s_i\lambda} + (q^{-1}-1)E^{s_i\sigma}_{s_i\lambda}$

$\because |\mathrm{Inv}(s_i\lambda)| = |\mathrm{Inv}(\lambda)|-1 < |\mathrm{Inv}(\lambda)|$

$\therefore$ By induction assumption, $E^{s_i\sigma}_{s_i\lambda}$ is monic (and triangular by def.)

is of this form

$\because T_i(x^{s_i\lambda} + \sum_{\mu<s_i\lambda} c_\mu x^\mu) = q x^{s_i(s_i\lambda)} + \sum_{\mu<s_i(s_i\lambda)} d_\mu x^\mu$ for some $d_\mu\in\mathbb{Q}(q)$

$= qx^\lambda + \sum_{\mu<\lambda} d_\mu x^\mu$

$\therefore q^{-1}T_iE^{s_i\sigma}_{s_i\lambda} = x^\lambda + \sum_{\mu<\lambda} q^{-1}d_\mu x^\mu$ ($\therefore E^\sigma_\lambda$ is monic if $s_i\sigma>\sigma$)

Note that $E^{s_i\sigma}_{s_i\lambda} = x^{s_i\lambda} + \sum_{\mu<s_i\lambda} c_\mu x^\mu$ and $\lambda>s_i\lambda$ when $\lambda_{i+1}>\lambda_i$

$\therefore E^{s_i\sigma}_{s_i\lambda}$ has no x^λ term

$\therefore q^{-1}T_iE^{s_i\sigma}_{s_i\lambda} + (q^{-1}-1)E^{s_i\sigma}_{s_i\lambda} = x^\lambda + (q^{-1}-1)x^{s_i\lambda} + \sum_{\mu<\lambda} q^{-1}d_\mu x^\mu + \sum_{\mu<s_i\lambda} c_\mu x^\mu = x^\lambda + \sum_{\mu<\lambda}\tilde{c}_\mu x^\mu$ for some $\tilde{c}_\mu\in\mathbb{Q}(q)$

Hence E^σ_λ is monic if $s_i\sigma<\sigma$.

$\therefore$ The statement is true by induction.

$\therefore$ The twisted non-symmetric polynomials E^σ_λ are monic and triangular, i.e. has the form $x^\lambda + \sum_{\mu<\lambda} c_\mu x^\mu$ for some $c_\mu\in\mathbb{Q}(q)$.

e.g. Compute $E^{s_1}_{479}$

Method I) Construct the non-twisted version

$\sigma=s_1 \Rightarrow s_1\sigma = id<\sigma$

Also, $\lambda_1<\lambda_2 \Rightarrow \mathbb{1}(\lambda_1\geq\lambda_2)=0$

$\because s_2s_1\cdot(974) = s_2\cdot(794) = 749$

$\therefore E_{749} = q^{-2}T_2T_1x^{974}$

$\therefore E^{s_1}_{479} = q^0T_1^{-1}E^{id}_{749} = (q^{-1}T_1 + q^{-1}-1)E_{749} = (q^{-1}T_1+q^{-1}-1)(q^{-2}T_2T_1x^{974}) = q^{-3}T_1T_2T_1x^{974} + (q^{-3}-q^{-2})T_2T_1x^{974}$

Method II) Make the subscript dominant and use $E^\sigma_{\lambda_+} = x^{\lambda_+}$

$s_1s_2s_1\cdot 479 = 974$,

$\lambda=479$: $s_1s_1 = id<s_1$ and $4<7 \Rightarrow \mathbb{1}(\lambda_1\geq\lambda_2)=0$ $\therefore E^{s_1}_{479} = q^0T_1^{-1}E^{id}_{749}$

$\lambda=749$: $s_2 id = s_2 > id$ and $4<9 \Rightarrow \mathbb{1}(\lambda_2\leq\lambda_3)=1$ $\therefore E^{s_1}_{479} = T_1^{-1}(q^{-1}T_2E^{s_2}_{794}) = q^{-1}T_1^{-1}T_2E^{s_2}_{794}$

$\lambda=794$: $s_1s_2>s_2$ and $7<9 \Rightarrow \mathbb{1}(\lambda_1\leq\lambda_2)=1$ $\therefore E^{s_1}_{479} = q^{-1}T_1^{-1}T_2(q^{-1}T_1E^{s_1s_2}_{974}) = q^{-2}T_1^{-1}T_2T_1x^{974} = q^{-2}(q^{-1}T_1+q^{-1}-1)T_2T_1x^{974} = (q^{-3}T_1T_2T_1 + (q^{-3}-q^{-2})T_2T_1)x^{974}$

$T_1x^{974} = (q-1)x^{974} + (q-1)x^{884} + qx^{794}$, $T_2T_1x^{974} = (q-1)^2(x^{974}+x^{965}+x^{956}+x^{884}+x^{875}+x^{866}+x^{857}) + q(q-1)(x^{947}+x^{848}+x^{794}+x^{785}+x^{776}+x^{767}+x^{758}) + q^2x^{749}$

$T_1T_2T_1x^{974} = (q-1)^3(x^{974}+x^{884}+x^{965}+x^{875}+x^{785}+x^{956}+x^{866}+x^{776}+x^{686}+x^{875}+x^{866}+x^{776}+x^{857}+x^{767}+x^{677}) + (q-1)^2q(x^{794}+x^{695}+x^{596}+x^{884}+x^{785}+x^{686}+x^{587}+x^{947}+x^{857}+x^{767}+x^{677}+x^{587}$
$+x^{848}+x^{758}+x^{668}+x^{578}-x^{884}+x^{767}+x^{758}+x^{668}) + q(q-1)(x^{974}+x^{875}) + q^2(q-1)(x^{497}+x^{488}+x^{776}+x^{677}+x^{578}+x^{749}+x^{659}+x^{569}) + q^3x^{479}$ (only x^{479} term)

$\therefore E^{s_1}_{479} = q^{-3}(q^3x^{479}) + \cdots = x^{479} + \cdots$ ← monic

Define an inner product on $\Bbbk[x_1^{\pm 1},\dots,x_\ell^{\pm 1}]$:

$$\langle f, g\rangle_q := \langle x^0\rangle\, fg \prod_{i<j} \frac{1-\frac{x_i}{x_j}}{1-q^{-1}\frac{x_i}{x_j}}.$$

Lemma: T_i are self-adjoint w.r.t. $\langle\, ,\rangle_q$.

Proof: Note that $T_i - q$ is self-adjoint iff q is self-adjoint.

It is because $\langle (T_i-q)f, g\rangle_q = \langle T_i f, g\rangle_q - q\langle f,g\rangle_q$

$\langle f, (T_i-q)g\rangle_q = \langle f, T_i g\rangle_q - q\langle f, g\rangle_q$

$\therefore\ \langle (T_i-q)f, g\rangle_q = \langle f, (T_i-q)g\rangle_q$ iff $\langle T_i f, g\rangle_q = \langle f, T_i g\rangle_q \quad \forall f,g\in \Bbbk[x_1^{\pm 1},\dots,x_\ell^{\pm 1}]$.

Note that

multiply $-\frac{x_i}{x_{i+1}}$ on both numerator and denominator, then factor out q in the numerator

$$T_i - q = q(s_i-1) + \frac{1-q}{1-\frac{x_{i+1}}{x_i}}(s_i-1) = \frac{1-q\frac{x_{i+1}}{x_i}}{1-\frac{x_{i+1}}{x_i}}(s_i-1) = q\,\frac{1-q^{-1}\frac{x_i}{x_{i+1}}}{1-\frac{x_i}{x_{i+1}}}(s_i-1)$$

$$\therefore\ \langle (T_i-q)f, g\rangle_q = \left\langle q\,\frac{1-q^{-1}\frac{x_i}{x_{i+1}}}{1-\frac{x_i}{x_{i+1}}}(s_i-1)f, g\right\rangle_q = \langle x^0\rangle\left(q\,\frac{1-q^{-1}\frac{x_i}{x_{i+1}}}{1-\frac{x_i}{x_{i+1}}}(s_i-1)fg\cdot\prod_{j<k}\frac{1-\frac{x_j}{x_k}}{1-q^{-1}\frac{x_j}{x_k}}\right) = q\langle x^0\rangle\big[(s_i f)g - fg\big]\prod_{\substack{1\le j<k\le \ell\\ (j,k)\ne(i,i+1)}}\frac{1-\frac{x_j}{x_k}}{1-q^{-1}\frac{x_j}{x_k}}$$

$$\langle f, (T_i-q)g\rangle_q = \left\langle f, q\,\frac{1-q^{-1}\frac{x_i}{x_{i+1}}}{1-\frac{x_i}{x_{i+1}}}(s_i-1)g\right\rangle_q = \langle x^0\rangle\left(q\,\frac{1-q^{-1}\frac{x_i}{x_{i+1}}}{1-\frac{x_i}{x_{i+1}}}f(s_i-1)g\cdot\prod_{j<k}\frac{1-\frac{x_j}{x_k}}{1-q^{-1}\frac{x_j}{x_k}}\right) = q\langle x^0\rangle\big[f(s_i g) - fg\big]\prod_{\substack{1\le j<k\le \ell\\ (j,k)\ne(i,i+1)}}\frac{1-\frac{x_j}{x_k}}{1-q^{-1}\frac{x_j}{x_k}}$$

Hence it suffices to show

$$\langle x^0\rangle\,(s_i f)g\prod_{\substack{1\le j<k\le \ell\\ (j,k)\ne(i,i+1)}}\frac{1-\frac{x_j}{x_k}}{1-q^{-1}\frac{x_j}{x_k}} = \langle x^0\rangle\, f(s_i g)\prod_{\substack{1\le j<k\le \ell\\ (j,k)\ne(i,i+1)}}\frac{1-\frac{x_j}{x_k}}{1-q^{-1}\frac{x_j}{x_k}}$$

Note that $s_i\left(\prod_{\substack{1\le j<k\le \ell\\ (j,k)\ne(i,i+1)}}\frac{1-\frac{x_j}{x_k}}{1-q^{-1}\frac{x_j}{x_k}}\right) = \prod_{\substack{1\le j<k\le \ell\\ (j,k)\ne(i,i+1)}}\frac{1-\frac{x_j}{x_k}}{1-q^{-1}\frac{x_j}{x_k}}$ and $\langle x^0\rangle(\phi(x_1,\dots,x_\ell)) = \langle x^0\rangle(s_i\phi(x_1,\dots,x_\ell))$

$\therefore\ \langle x^0\rangle(s_i f)g\prod_{\substack{1\le j<k\le \ell\\ (j,k)\ne(i,i+1)}}\frac{1-\frac{x_j}{x_k}}{1-q^{-1}\frac{x_j}{x_k}} = \langle x^0\rangle\, s_i\left((s_i f)g\prod_{\substack{1\le j<k\le \ell\\ (j,k)\ne(i,i+1)}}\frac{1-\frac{x_j}{x_k}}{1-q^{-1}\frac{x_j}{x_k}}\right) = \langle x^0\rangle\, f(s_i g)\prod_{\substack{1\le j<k\le \ell\\ (j,k)\ne(i,i+1)}}\frac{1-\frac{x_j}{x_k}}{1-q^{-1}\frac{x_j}{x_k}}$ and result follows.

$\square$

Prop: For any $\sigma\in S_\ell$, $\{E_\lambda^\sigma\}_{\lambda\in\mathbb{Z}^\ell}$ and $\{\overline{F_\lambda^\sigma}\}_{\lambda\in\mathbb{Z}^\ell}$ are dual bases of $\Bbbk[x_1^{\pm 1},\dots,x_\ell^{\pm 1}]$ w.r.t $\langle\, ,\rangle_q$.

i.e. $\langle E_\lambda^\sigma, \overline{F_\mu^\sigma}\rangle_q = \delta_{\lambda\mu} \quad \forall\lambda,\mu\in\mathbb{Z}^\ell,\ \sigma\in S_\ell$.

Proof: It is equivalent to show $\langle E_\lambda^\sigma, E_{-\mu}^{\sigma w_0}\rangle_q = \delta_{\lambda\mu}$. (Note: They are bases b/c E_λ^σ has a monic triangular form and F_λ^σ are "basically" E_λ^σ's with $x_i\to x_i^{-1}$, $q\to q^{-1}$)

$$\langle E_\lambda^\sigma, E_{-\mu}^{\sigma w_0}\rangle_q = \begin{cases} q^{-\mathbb{1}(\lambda_i\le\lambda_{i+1})+\mathbb{1}(-\mu_i\ge-\mu_{i+1})}\ \langle T_i E_{s_i\lambda}^{s_i\sigma}, T_i^{-1}E_{s_i(-\mu)}^{s_i\sigma w_0}\rangle_q & \text{if } s_i\sigma>\sigma\\ q^{\mathbb{1}(\lambda_i\ge\lambda_{i+1})-\mathbb{1}(-\mu_i\le\mu_{i+1})}\ \langle T_i^{-1}E_{s_i\lambda}^{s_i\sigma}, T_i E_{s_i(-\mu)}^{s_i\sigma w_0}\rangle_q & \text{if } s_i\sigma<\sigma\end{cases}$$

Since T_i is self-adjoint w.r.t. $\langle\, ,\rangle_q$, we have

$$\langle E_\lambda^\sigma, E_{-\mu}^{\sigma w_0}\rangle_q = \begin{cases} q^{-\mathbb{1}(\lambda_i\le\lambda_{i+1})+\mathbb{1}(\mu_i\le\mu_{i+1})}\ \langle E_{s_i\lambda}^{s_i\sigma}, E_{s_i(-\mu)}^{s_i\sigma w_0}\rangle_q & \text{if } s_i\sigma>\sigma\\ q^{\mathbb{1}(\lambda_i\ge\lambda_{i+1})-\mathbb{1}(\mu_i\ge\mu_{i+1})}\ \langle E_{s_i\lambda}^{s_i\sigma}, E_{s_i(-\mu)}^{s_i\sigma w_0}\rangle_q & \text{if } s_i\sigma<\sigma\end{cases} \quad \text{—— } (*)$$

Note that if $\lambda=\mu$, then $-\mathbb{1}(\lambda_i\le\lambda_{i+1})+\mathbb{1}(\mu_i\le\mu_{i+1}) = \mathbb{1}(\lambda_i\ge\lambda_{i+1})-\mathbb{1}(\mu_i\ge\mu_{i+1}) = 0$

$$\langle E_\lambda^\sigma, E_{-\lambda}^{\sigma w_0}\rangle_q = \begin{cases}\langle E_{s_i\lambda}^{s_i\sigma}, E_{s_i(-\lambda)}^{s_i\sigma w_0}\rangle_q & \text{if } s_i\sigma>\sigma\\ \langle E_{s_i\lambda}^{s_i\sigma}, E_{s_i(-\lambda)}^{s_i\sigma w_0}\rangle_q & \text{if } s_i\sigma<\sigma\end{cases}$$

Repeat the recurrence $(*)$, we get

$\langle E_\lambda^\sigma, E_{-\mu}^{\sigma w_0}\rangle_q = q^e\,\langle E_{v(\lambda)}^{v\sigma}, E_{\underbrace{v(-\mu)}_{-v(\mu)}}^{v\sigma w_0}\rangle_q$ for some $e\in\mathbb{Z}$ and $e=0$ for $\lambda=\mu$.

Take $\nu \in S_\ell$ s.t. $\nu(\mu) = \mu_-$. Then $E^{\nu\sigma w_0}_{-\nu(\mu)} = E^{\nu\sigma w_0}_{\underbrace{-\mu_-}_{\because \text{Dominant}}} = x^{-\mu_-}$

$$\langle E^\sigma_\lambda, E^{\sigma w_0}_{-\mu} \rangle_q = q^e \langle E^{\nu\sigma}_{\nu(\lambda)}, x^{-\mu_-} \rangle_q = q^e \langle x^0 \rangle E^{\nu\sigma}_{\nu(\lambda)} \cdot x^{\mu_-} \prod_{1\le j<k\le \ell} \frac{1-\frac{x_j}{x_k}}{1-q^{-1}\frac{x_j}{x_k}} = q^e \langle x^{\mu_-} \rangle E^{\nu\sigma}_{\nu(\lambda)} \prod_{1\le j<k\le \ell} \frac{1-\frac{x_j}{x_k}}{1-q^{-1}\frac{x_j}{x_k}}$$

Notation: $\mathrm{supp}(f) = \{ \gamma \in \mathbb{Z}_\ell : \text{coeff of } x^\gamma \text{ in } f \ne 0 \}$

$$\because \prod_{1\le j<k\le \ell} \frac{1-\frac{x_j}{x_k}}{1-q^{-1}\frac{x_j}{x_k}} = \prod_{1\le j<k\le \ell} \left(1-\frac{x_j}{x_k}\right)\left(1+q^{-1}\frac{x_j}{x_k}+q^{-2}\frac{x_j^2}{x_k^2}+\cdots\cdots\right)$$

$$= \prod_{1\le j<k\le \ell} \left[1+(q^{-1}-1)\frac{x_j}{x_k}+(q^{-2}-q^{-1})\frac{x_j^2}{x_k^2}+(q^{-3}-q^{-2})\frac{x_j^3}{x_k^3}+\cdots\cdots\right]$$

$$= \prod_{1\le j<k\le \ell} \left[1+q^{-1}(1-q)\frac{x_j}{x_k}+q^{-2}(1-q)\frac{x_j^2}{x_k^2}+q^{-3}(1-q)\frac{x_j^3}{x_k^3}+\cdots\right]$$

$$= 1+\sum_{\alpha\in Q_+\setminus\{0\}} c_\alpha(q)\, x^\alpha \quad \text{where } c_\alpha(q)\ne 0 \text{ and } c_\alpha(q)\in\mathbb{Q}(q) \quad \text{is divisible by } 1-q$$

$$\therefore \mathrm{supp}\left(\prod_{1\le j<k\le \ell} \frac{1-\frac{x_j}{x_k}}{1-q^{-1}\frac{x_j}{x_k}}\right) = Q_+.$$

We know $\mathrm{supp}(E^{\nu\sigma}_{\nu(\lambda)}) \subseteq \mathrm{conv}(S_\ell\cdot\lambda)$, hence if $\langle E^\sigma_\lambda, E^{\sigma w_0}_{-\mu}\rangle_q \ne 0$, i.e. $q^e \langle x^{\mu_-}\rangle E^{\nu\sigma}_{\nu(\lambda)} \prod_{1\le j<k\le \ell} \frac{1-\frac{x_j}{x_k}}{1-q^{-1}\frac{x_j}{x_k}} \ne 0$, then

$\mu_- \in \mathrm{supp}\left(E^{\nu\sigma}_{\nu(\lambda)} \cdot \prod_{1\le j<k\le \ell} \frac{1-\frac{x_j}{x_k}}{1-q^{-1}\frac{x_j}{x_k}}\right)$.

$$\because E^{\nu(\sigma)}_{\nu(\lambda)} \prod_{1\le j<k\le \ell} \frac{1-\frac{x_j}{x_k}}{1-q^{-1}\frac{x_j}{x_k}} = \sum_{\gamma\in\mathrm{conv}(S_\ell\cdot\lambda)} d_\gamma x^\gamma \cdot \sum_{\alpha\in Q_+} c_\alpha x^\alpha \quad \text{where } c_0=1,\ c_\alpha, d_\alpha \in \mathbb{Q}(q)$$

$$= \sum_{\substack{\gamma\in\mathrm{conv}(S_\ell\cdot\lambda)\\ \alpha\in Q_+}} d_\gamma c_\alpha x^{\gamma+\alpha}$$

$\therefore \mu_- = \gamma+\alpha$ for some $\gamma\in\mathrm{conv}(S_\ell\cdot\lambda)$ and $\alpha\in Q_+$ s.t. $d_\gamma, c_\alpha \ne 0$

i.e. $\mu_- - \alpha = \gamma \in \mathrm{conv}(S_\ell\cdot\lambda)$

$\therefore (\mu_- - Q_+) \cap \mathrm{conv}(S_\ell\cdot\lambda) \ne \emptyset$

$\because \gamma - \lambda_- \in Q_+$ for $\gamma\in\mathrm{conv}(S_\ell\cdot\lambda)$

$\therefore \mu_- - \lambda_- = \mu_- - \gamma + \gamma - \lambda_- = \alpha + (\gamma-\lambda_-) \in Q_+$

$\because w_0(Q_+) = -Q_+$

$\therefore \mu_+ - \lambda_+ \in -Q_+ \Rightarrow \lambda_+ - \mu_+ \in Q_+ \Rightarrow \lambda_+ \geqslant \mu_+$.

$\therefore \langle E^\sigma_\lambda, E^{\sigma w_0}_{-\mu}\rangle_q \ne 0 \Rightarrow \lambda_+ \geqslant \mu_+$

Also, $\langle E^{\sigma w_0}_{-\mu}, E^\sigma_\lambda \rangle_q = \langle E^{\sigma w_0}_{-\mu}, E^{\sigma w_0 w_0}_{-(-\lambda)} \rangle_q$

$\therefore \langle E^{\sigma w_0}_{-\mu}, E^\sigma_\lambda\rangle_q \ne 0 \Rightarrow (-\mu)_+ \geqslant (-\lambda)_+ \Leftrightarrow -\mu_- \geqslant -\lambda_- \Leftrightarrow \lambda_- \geqslant \mu_- \Leftrightarrow \mu_+ \geqslant \lambda_+$

$\because \langle E^{\sigma w_0}_{-\mu}, E^\sigma_\lambda\rangle_q = \langle E^\sigma_\lambda, E^{\sigma w_0}_{-\mu}\rangle_q$

$\therefore \langle E^\sigma_\lambda, E^{\sigma w_0}_{-\mu}\rangle_q \ne 0 \Rightarrow \lambda_+ = \mu_+$ and hence $S_\ell\cdot\lambda = S_\ell\cdot\mu$

$\therefore$ We only need to consider λ, μ s.t. $S_\ell \cdot \lambda = S_\ell \cdot \mu$.

In this case, $(\mu_- - Q_+) \cap \operatorname{conv}(S_\ell \cdot \lambda) = \{\mu_-\}$ b/c $\mu_- - \alpha \notin \operatorname{conv}(S_\ell \cdot \mu)$ $\forall \alpha \in Q_+ \setminus \{0\}$

If $\lambda \neq \mu$, then $v(\lambda) \neq v(\mu)$ i.e. $v(\lambda) \neq \mu_-$.

$\because E^{v\sigma}_{v(\lambda)} = x^{v(\lambda)} + \sum_{\gamma < v(\lambda)} c_\gamma x^\gamma$ and $\mu_- = \lambda_- > v(\lambda)$

$\therefore \mu_- \notin \operatorname{supp}(E^{v\sigma}_{v(\lambda)}) \subseteq \operatorname{conv}(S_\ell \cdot \lambda)$

Hence $(\mu_- - Q_+) \cap \operatorname{supp}(E^{v(\sigma)}_{v(\lambda)}) = \phi$. Recall

$$\because \ E^{v(\sigma)}_{v(\lambda)} \prod_{1 \le j < k \le \ell} \frac{1 - \frac{x_j}{x_k}}{1 - q^{-1}\frac{x_j}{x_k}} = \sum_{\substack{\gamma \in \operatorname{conv}(S_\ell \cdot \lambda) \\ \alpha \in Q_+}} d_\gamma c_\alpha x^{\gamma + \alpha}$$

$$\therefore \ \mu_- \in \operatorname{supp}\left(E^{v(\sigma)}_{v(\lambda)} \prod_{1 \le j < k \le \ell} \frac{1 - \frac{x_j}{x_k}}{1 - q^{-1}\frac{x_j}{x_k}} \right) \Rightarrow \mu_- = \gamma + \alpha \text{ for some } \gamma \in \operatorname{conv}(S_\ell \cdot \lambda) \text{ and } \alpha \in Q_+ \text{ s.t. } d_\gamma, c_\alpha \neq 0$$

$$\text{i.e. } \mu_- - \alpha = \gamma \underset{\substack{\uparrow \\ \text{b/c } d_\gamma \neq 0}}{\in} \operatorname{supp}(E^{v(\sigma)}_{v(\lambda)}) \Rightarrow (\mu_- - Q) \cap \operatorname{supp}(E^{v(\sigma)}_{v(\lambda)}) \neq \phi \text{ if } \mu_- \in \operatorname{supp}\left(E^{v(\sigma)}_{v(\lambda)} \prod_{1 \le j < k \le \ell} \frac{1 - \frac{x_j}{x_k}}{1 - q^{-1}\frac{x_j}{x_k}} \right).$$

$$\therefore (\mu_- - Q_+) \cap \operatorname{supp}(E^{v(\sigma)}_{v(\lambda)}) = \phi \Rightarrow \mu_- \notin \operatorname{supp}\left(E^{v(\sigma)}_{v(\lambda)} \prod_{1 \le j < k \le \ell} \frac{1 - \frac{x_j}{x_k}}{1 - q^{-1}\frac{x_j}{x_k}} \right) \Rightarrow \langle E^\sigma_\lambda, E^{\sigma w_0}_{-\mu} \rangle_q = 0$$

Hence $\lambda \neq \mu \Rightarrow \langle E^\sigma_\lambda, E^{\sigma w_0}_{-\mu} \rangle_q = 0$

If $\lambda = \mu$, then $\langle E^\sigma_\lambda, E^{\sigma w_0}_{-\lambda} \rangle_q = \langle x^{\lambda_-} \rangle E^{v(\sigma)}_{v(\lambda)} \prod_{1 \le j < k \le \ell} \frac{1 - \frac{x_j}{x_k}}{1 - q^{-1}\frac{x_j}{x_k}} = \langle x^{\lambda_-} \rangle E^{v(\sigma)}_{\lambda_-} \prod_{1 \le j < k \le \ell} \frac{1 - \frac{x_j}{x_k}}{1 - q^{-1}\frac{x_j}{x_k}}$

$\because \operatorname{supp}(E^{v(\sigma)}_{\lambda_-}) \subset \lambda_- + Q_+$ and $\operatorname{supp}\left(\prod_{1 \le j < k \le \ell} \frac{1 - \frac{x_j}{x_k}}{1 - q^{-1}\frac{x_j}{x_k}} \right) = Q_+$

$\therefore$ The only way to form x^{λ_-} in $E^{v(\sigma)}_{\lambda_-} \cdot \prod_{1 \le j < k \le \ell} \frac{1 - \frac{x_j}{x_k}}{1 - q^{-1}\frac{x_j}{x_k}}$ is taking the product of the x^{λ_-} term in $E^{v(\sigma)}_{\lambda_-}$ and the constant term in $\prod_{1 \le j < k \le \ell} \frac{1 - \frac{x_j}{x_k}}{1 - q^{-1}\frac{x_j}{x_k}}$.

i.e. coeff. of x^{λ_-} in $E^{v(\sigma)}_{\lambda_-} \cdot \prod_{1 \le j < k \le \ell} \frac{1 - \frac{x_j}{x_k}}{1 - q^{-1}\frac{x_j}{x_k}}$ is the product of the coeff. of x^{λ_-} in $E^{v(\sigma)}_{\lambda_-}$ and the constant term in $\prod_{1 \le j < k \le \ell} \frac{1 - \frac{x_j}{x_k}}{1 - q^{-1}\frac{x_j}{x_k}}$.

$\because E^{v(\sigma)}_{\lambda_-} = x^{\lambda_-} + \sum_{\gamma < \lambda_-} c_\gamma x^\gamma$ for some $c_\gamma \in \mathbb{Q}(q)$ and $\prod_{1 \le j < k \le \ell} \frac{1 - \frac{x_j}{x_k}}{1 - q^{-1}\frac{x_j}{x_k}} = 1 + \sum_{\alpha \in Q_+ \setminus \{0\}} c_\alpha(q) x^\alpha$ whose constant term is 1

$\therefore$ coeff of x^{λ_-} in $E^{v(\sigma)}_{\lambda_-} \cdot \prod_{1 \le j < k \le \ell} \frac{1 - \frac{x_j}{x_k}}{1 - q^{-1}\frac{x_j}{x_k}} = 1 \times 1 = 1$

i.e. $\langle E^\sigma_\lambda, E^{\sigma w_0}_{-\lambda} \rangle_q = \langle x^{\lambda_-} \rangle E^{v(\sigma)}_{\lambda_-} \prod_{1 \le j < k \le \ell} \frac{1 - \frac{x_j}{x_k}}{1 - q^{-1}\frac{x_j}{x_k}} = 1$.

As a result, $\langle E^\sigma_\lambda, E^{\sigma w_0}_{-\mu} \rangle_q = \delta_{\lambda\mu}$.

$\square$

Lemma: Let $\lambda \in \mathbb{Z}^\ell$. Suppose $\exists k \in [\ell]$ s.t. $\lambda_i \geq \lambda_j$ $\forall i \le k$ and $j > k$.

Given $\sigma \in S_\ell$, define $\sigma_1 \in S_k$ and $\sigma_2 \in S_{\ell-k}$ s.t. $\sigma_1(1), \ldots, \sigma_1(k)$ are in the same relative order as $\sigma(1), \ldots, \sigma(k)$, and $\sigma_2(1), \ldots, \sigma_2(\ell-k)$ are in the same relative order as $\sigma(k+1), \ldots, \sigma(\ell)$. Then

$$E^{\sigma^{-1}}_\lambda(x_1, \ldots, x_\ell; q) = E^{\sigma_1^{-1}}_{(\lambda_1, \ldots, \lambda_k)}(x_1, \ldots, x_k; q)\, E^{\sigma_2^{-1}}_{(\lambda_{k+1}, \ldots, \lambda_\ell)}(x_{k+1}, \ldots, x_\ell; q).$$

Proof: If $\lambda = \lambda_+$, then $E^{\sigma^{-1}}_\lambda = x^\lambda = (x_1^{\lambda_1} \cdots x_k^{\lambda_k}) \cdot (x_{k+1}^{\lambda_{k+1}} \cdots x_\ell^{\lambda_\ell}) = E^{\sigma_1^{-1}}_{(\lambda_1, \ldots, \lambda_k)}(x_1, \ldots, x_k; q)\, E^{\sigma_2^{-1}}_{(\lambda_{k+1}, \ldots, \lambda_\ell)}(x_{k+1}, \ldots, x_\ell; q)$. Hence the statement is trivial for $\lambda = \lambda_+$.

k can be any index in this case.

Otherwise, $E^{\sigma^{-1}}_\lambda$ can be constructed by the recurrence

$$E^\sigma_\lambda = \begin{cases} q^{-\mathbb{1}(\lambda_{i+1} \geq \lambda_i)} T_i E^{s_i\sigma}_{s_i\lambda}(x; q) & \text{if } s_i\sigma > \sigma \\ q^{\mathbb{1}(\lambda_i \geq \lambda_{i+1})} T_i^{-1} E^{s_i\sigma}_{s_i\lambda}(x; q) & \text{if } s_i\sigma < \sigma \end{cases}$$

on induction on $|\mathrm{Inv}(-\lambda)| = |\{(a,b) : a<b \text{ and } \lambda_a < \lambda_b\}|$

Hence we only need to consider $1 \le a < b \le k$ and $k < a < b \le \ell$.

$\therefore$ We only take i s.t. $s_i \in S_k \times S_{\ell-k} \subset S_\ell$ when using the recurrence.

$\because$ Both $E^{\sigma_1^{-1}}_{(\lambda_1,\dots,\lambda_k)}(x_1,\dots,x_k;q)$ and $E^{\sigma_2^{-1}}_{(\lambda_{k+1},\dots,\lambda_\ell)}(x_{k+1},\dots,x_\ell;q)$ satisfy the same recurrence and the recurrence determines $E^{\sigma^{-1}}_\lambda$, result follows. $\square$

e.g. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 7 & 1 & 4 & 8 & 3 & 2 & 6 \end{pmatrix}$, $\lambda = (6,9,5,6,7,1,3,2)$. Hence $k=5$.

$\therefore \sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

Hence $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 6 & 4 & 1 & 8 & 2 & 5 \end{pmatrix}$, $\sigma_1^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}$, $\sigma_2^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. By the lemma above, we have

$$E^{\sigma^{-1}}_{69567132}(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8;q) = E^{\sigma_1^{-1}}_{69567}(x_1,x_2,x_3,x_4,x_5;q)\, E^{\sigma_2^{-1}}_{132}(x_6,x_7,x_8;q).$$