

Def: The non-symmetric Hall-Littlewood polynomial is defined as

$$E_\lambda(x; q) = E_\lambda(x_1, \dots, x_l; q) := \sum_{\mu \in \mathbb{Z}^l} c_{\mu\lambda} T_\mu x^\lambda$$

where $\lambda \in \mathbb{Z}^l$ (GL weight) and $\mu \in \mathbb{Z}^l$ s.t. $\lambda = \omega(\lambda_+)$ (ω is not unique if λ has equal parts, but E_λ is independent of the choice of ω b/c $T_i x^\lambda = q x^\lambda$ for $i \in \lambda$ and the extra q factor is canceled by $q^{-\text{len}(\lambda)}$ if ω has a longer length)

Remark: $E_\lambda(x; q)$ is monic and triangular:

Also, $c_{\mu\lambda} \in \mathbb{Q}(q)$

$$E_\lambda(x; q) = x^\lambda + \sum_{\mu < \lambda} c_{\mu\lambda} x^\mu \quad (\text{Recall } T_i(x^\lambda + \sum_{\mu < \lambda} c_{\mu\lambda} x^\mu) = q x^\lambda + \sum_{\mu < \lambda} c_{\mu\lambda} x^\mu \forall \mu \in \mathbb{Z}^l. \text{ Thus, } E_\lambda(x; q) = \sum_{\mu < \lambda} c_{\mu\lambda} T_\mu x^\lambda = q^{\text{len}(\lambda)} \left(\sum_{\mu < \lambda} c_{\mu\lambda} x^\mu \right) + \sum_{\mu < \lambda} d_{\mu\lambda} x^\mu = x^\lambda + \sum_{\mu < \lambda} d_{\mu\lambda} x^\mu)$$

e.g. $\lambda = (4, 9, 7), l=3 \Rightarrow \lambda_+ = (9, 7, 4), \omega = S_2$ check: $S_2 \cdot \lambda_+ = (9, 7, 4) \leq \lambda_+ = \omega \cdot (9, 7, 4) = (4, 9, 7) = \lambda$
 $\therefore E_{(4,9,7)}(x_1, x_2, x_3; q) = \sum_{\mu < \lambda} c_{\mu\lambda} T_\mu x^\mu$

$$= q^2 T_1 (q-1) x^{974} + (q-1) x^{965} + (q-1) x^{956} + q x^{947}$$

$$= q^2 T_1 (q-1)^2 (x^{974} + x^{884} + x^{965} + x^{875} + x^{956} + x^{866} + x^{776} + x^{686}) + q^2 (q-1) (x^{764} + x^{695} + x^{596} + x^{497} + x^{398} + x^{299} + x^{190} + x^{091})$$

$$+ q^2 [(q-1) x^{947} + (q-1) x^{857} + (q-1) x^{767} + (q-1) x^{677} + (q-1) x^{587} + (q-1) x^{497}]$$

$$= q^2 (q-1)^2 (x^{974} + x^{884} + x^{965} + x^{875} + x^{956} + x^{866} + x^{776} + x^{686}) + q^2 (q-1) (x^{764} + x^{695} + x^{596} + x^{497} + x^{398} + x^{299} + x^{190} + x^{091})$$

$$+ x^{497} \quad \text{all } \mu \text{ in } x^\mu \text{ satisfy } \mu < 497$$

$$= x^{497} + (-q^2) x^{764} + x^{695} + x^{596} + x^{497} + x^{398} + x^{299} + x^{190} + x^{091} + (-q^2) x^{974} + x^{884} + x^{965} + x^{875} + x^{956} + x^{866} + x^{776} + x^{686}$$

$$\uparrow \text{monic} \quad \uparrow \text{in } \mathbb{Q}[q^{\pm 1}]$$

(F-version)

$$F_\lambda(x; q) = F_\lambda(x_1, \dots, x_l; q) := T_\lambda x^\lambda$$

where $\lambda \in \mathbb{Z}^l$ (GL weight), λ_- : weakly decreasing parts of λ (i.e. $\lambda_- = \omega_0(\lambda_+)$) and $\omega \in S_l$ s.t. $\lambda = \omega(\lambda_-)$

longest permutation in S_l

e.g. $\lambda = (4, 9, 7), l=3 \Rightarrow \lambda_- = (4, 7, 9), \omega = S_2$

$$\therefore F_{(4,9,7)}(x; q) = T_2(x^{497}) = (1-q) x^{488} + x^{497}$$

↑ coefficient

Remark: $q \rightarrow \infty$: E_λ are Demazure characters / key polynomials

$q \rightarrow 0$: F_λ are Demazure atoms (may need to reverse the variables)

Recall $\text{Inv}(\sigma) = \{(i, j) : i \leq j \text{ and } \sigma(i) > \sigma(j)\}$ for any $\sigma \in S_l$. We extend this to \mathbb{R}^l : $\text{Inv}(\mu) := \{(i, j) : i \leq j, \mu_i > \mu_j\}, \mu \in \mathbb{R}^l$.

Take p s.t. $\langle \alpha_i, p \rangle = 1$ (say $p = (1, -1, \dots, 1, -1)$) and take $\epsilon > 0$ (small), then

$$\text{Inv}(\lambda + \epsilon p) = \{(i, j) : i \leq j \text{ and } \lambda_i > \lambda_j\}.$$

$$\uparrow \text{b/c } (\lambda + \epsilon p)_j > (\lambda + \epsilon p)_i \Leftrightarrow \lambda_i - \lambda_j > \epsilon(p_j - p_i) = \epsilon(\lambda_i - \lambda_j) \text{ and } \epsilon > 0 \text{ small} \Rightarrow \lambda_i - \lambda_j \geq 0$$

Def: Let $\sigma \in S_l$. The twisted non-symmetric Hall-Littlewood polynomials is defined as

$$\bullet E_\lambda^\sigma(x; q) = \sum_{\substack{\text{Inv}(\sigma) \cap \text{Inv}(\lambda + \epsilon p)}} T_{\sigma^{-1}}^{-1} E_{\sigma^{-1}\lambda}(x; q) \quad (\text{since } T_i^{-1} = \frac{1}{q} T_i + (\frac{1}{q} - 1) \text{ and } E_\mu \text{ is triangular } \forall \mu, E_\lambda^\sigma \text{ is also triangular})$$

b/c T_i acts on triangular expressions gives triangular expressions

$$\bullet F_\lambda^\sigma(x; q) = \overline{E_\lambda^\sigma(x; q)} = \overline{E_\lambda^\sigma(x_1, x_2, \dots, x_l; q)}$$

where $\omega(l) = l+1-i \quad \forall 1 \leq i \leq l$.

Suppose $s: \sigma > \sigma$, then

$$E_{s\sigma}^\sigma(x; q) = \frac{1}{q} \text{Inv}(\sigma^s) \cap \text{Inv}(\sigma + \epsilon p)$$

$$\uparrow T_{(s\sigma)^{-1}}^{-1} E_{(s\sigma)^{-1}\sigma}(x; q)$$

$$\uparrow \sigma \cdot \lambda$$

$$\uparrow (T_i^{-1} T_j)^{-1}$$

$$\left\{ \begin{array}{l} (i, j) \in \text{Inv}(\sigma) \Leftrightarrow (i, j) \in \text{Inv}(\sigma^s) \\ ((i, j) \in \text{Inv}(\sigma) \Leftrightarrow (i, j) \in \text{Inv}(\sigma^s)) \cap ((j, i) \in \text{Inv}(\sigma) \Leftrightarrow (j, i) \in \text{Inv}(\sigma^s)) \\ (j, i) \text{ is inversed} \Leftrightarrow (j, i) \in \text{Inv}(\sigma^s) \end{array} \right\} \Rightarrow \text{Inv}(\sigma^s) = s \text{ Inv}(\sigma) \sqcup \{ (i, i) \}$$

$$|\text{Inv}(\sigma^s)| = |\text{Inv}(\sigma)| + |\text{Inv}(\sigma + \epsilon p)| = |\text{Inv}(\sigma)| + |\text{Inv}(\sigma + \epsilon p)|$$

$$= |\text{Inv}(\sigma)| + |\text{Inv}(\lambda + \epsilon p)| + |\{ (i, i) \}|$$

$$= |\text{Inv}(\sigma)| + |\text{Inv}(\lambda + \epsilon p)| + |\{ (i, i) \}|$$

$$\therefore E_{S,\lambda}^{\sigma}(x; q) = \frac{q^{(\text{Inv}(\sigma) \cap \text{Inv}(\lambda + \epsilon\sigma)) + (\text{Inv}(\sigma) \cap \lambda)}}{q} T_i^{-1} T_{\sigma}^{-1} E_{S,\lambda}^{\sigma}(x; q)$$

$$= \frac{q^{(\lambda_i > \lambda_j)}}{q} T_i^{-1} E_{\lambda}^{\sigma}(x; q)$$

$$\text{Hence } E_{\lambda}^{\sigma}(x; q) = \frac{q^{-\text{Inv}(\lambda)}}{q} T_i^{-1} E_{S,\lambda}^{\sigma}(x; q) \text{ if } S;\sigma > \sigma$$

$$\text{Similarly, } E_{\lambda}^{\sigma}(x; q) = \frac{q^{(\lambda_i > \lambda_{j+1})}}{q} T_i^{-1} E_{S,\lambda}^{\sigma}(x; q)$$

Treat $S;\sigma$ as σ

S,λ as λ

Then $\lambda_{j+1} > \lambda_i$ becomes $\lambda_{j+1} > \lambda_i$
b/c it means $(\lambda_i)_{i \in S} > (\lambda_j)_{j \in S}$:
i.e. $\lambda_i > \lambda_{j+1}$

Thus we have the recurrence:

$$E_{\lambda}^{\sigma} = \begin{cases} q^{-\text{Inv}(\sigma)} T_i^{-1} E_{S,\lambda}^{\sigma}(x; q) & \text{if } S;\sigma > \sigma \\ q^{(\lambda_i > \lambda_{j+1})} T_i^{-1} E_{S,\lambda}^{\sigma}(x; q) & \text{if } S;\sigma < \sigma \end{cases}$$

$$\text{Note: (1) } E_{\lambda}^{\sigma} = x^{\lambda_+} \quad \forall \sigma \in S_2$$

$$(2) \text{ If } \sigma = \text{id}, \text{ then } E_{\lambda}^{\sigma} = E_{\lambda}, F_{\lambda}^{\sigma} = F_{\lambda}.$$

$$\text{Check: } \frac{q^{(\text{Inv}(\sigma) \cap \text{Inv}(\lambda + \epsilon\sigma))}}{q} = q^{(\text{Inv}(\sigma))} = q^{l(\sigma^{-1})} \quad b/c \text{ Inv}(\lambda + \epsilon\sigma) = \{(i,j) : 1 \leq i < j \leq l\}$$

$$\cdot E_{\lambda^+}^{\sigma}(x; q) = \frac{q^{-l(\sigma)}}{q} T_w x^{\lambda_+} \quad \text{where } ad(\sigma^{-1}\lambda)_+ = \sigma^{-1}\lambda_+ \Rightarrow w = \sigma^{-1}$$

$$= q^{-l(\sigma^{-1})} T_{\sigma^{-1}} x^{\lambda_+}$$

$$\therefore E_{\lambda}^{\sigma} = \frac{q^{(\text{Inv}(\sigma) - l(\sigma))}}{q} T_{\sigma^{-1}} T_{\sigma^{-1}} x^{\lambda_+} = x^{\lambda_+} \text{ which proves (1).}$$

$$\text{Since } \sigma^{-1} = \text{id} \text{ for } \sigma = \text{id}, \text{ Inv}(\sigma^{-1}) = \emptyset. \text{ Hence } E_{\lambda}^{\text{id}} = \frac{q}{q} T_{\text{id}}^{-1} E_{\text{id},\lambda} = E_{\lambda}$$

$$F_{\lambda}^{\text{id}} = E_{\lambda}^{w_0}(x_1^{-1}, x_2^{-1}, \dots, x_k^{-1}; q^{-1}) \quad \because \text{Inv}(w_0^{-1}) = \text{Inv}(w_0) = \{(i,j) : 1 \leq i < j \leq l\}$$

$$\therefore \text{Inv}(w_0^{-1}) \cap \text{Inv}(-\lambda + \epsilon\text{id}) = \text{Inv}(-\lambda + \epsilon\text{id}) = \{(i,j) : i < j, -\lambda_i > -\lambda_j\} = \{(i,j) : i < j, \lambda_i < \lambda_j\}$$

$$\therefore E_{\lambda}^{w_0}(x_1^{-1}, x_2^{-1}, \dots, x_k^{-1}; q^{-1}) = \frac{q^{-l(\text{Inv}(-\lambda + \epsilon\text{id}))}}{q} T_{w_0}^{-1} E_{-\lambda}^{\text{id}}(x; q)$$

$$= \frac{q^{-l(\text{Inv}(-\lambda + \epsilon\text{id}))}}{q} T_{w_0}^{-1} \left(\frac{q^{-l(w)}}{q} T_w x^{\lambda_+} \right)$$

$$\text{where } w_0 \cdot (-\lambda) = w_0(-\lambda)_+ = -w_0(\lambda_-) \Rightarrow w_0(\lambda) = w(\lambda_-) \Rightarrow \boxed{\lambda = w_0 w(\lambda_-)} \Rightarrow l(w) + l(w_0 w) = \binom{l}{2}$$

$$\text{Inv}(-\lambda + \epsilon\text{id}) = \{(i,j) : i < j, (-\lambda)_i \geq (-\lambda)_j\} = \{(i,j) : i < j, \lambda_i \leq \lambda_j\} \quad \therefore w_0 \text{ should not change position } (i,j)$$

$$\therefore l(w) = \text{Inv}(-\lambda + \epsilon\text{id})$$

$$\therefore E_{-\lambda}^{w_0}(x_1^{-1}, x_2^{-1}, \dots, x_k^{-1}; q^{-1}) = T_{w_0}^{-1} T_w x^{\lambda_+}$$

$$\text{Note that } \begin{array}{c} \overbrace{\hspace{1cm}}^{w_0 w} \overbrace{\hspace{1cm}}^{w^{-1}} \\ \overbrace{\hspace{1cm}}^{w_0} \end{array} \Rightarrow \begin{array}{c} \overbrace{\hspace{1cm}}^{w_0 w} \overbrace{\hspace{1cm}}^{w^{-1}} \overbrace{\hspace{1cm}}^{w} \overbrace{\hspace{1cm}}^{(w_0 w)^{-1}} \\ \overbrace{\hspace{1cm}}^{w_0} \overbrace{\hspace{1cm}}^{w^{-1}} \overbrace{\hspace{1cm}}^{w} \overbrace{\hspace{1cm}}^{w_0} \end{array}$$

$$T_w T_{w_0}^{-1} T_w = T_{w_0 w}$$

$$\therefore F_{\lambda}^{\text{id}} = E_{-\lambda}^{w_0}(x_1^{-1}, x_2^{-1}, \dots, x_k^{-1}; q^{-1}) = T_{w_0 w} x^{\lambda_+} = F_{\lambda}$$

By (1) and recurrence, we can determine E_{λ}^{σ} $\forall \sigma$ and λ .

Note that $E_{\lambda}^{\sigma} = x^{\lambda}$ which is monic $\forall \sigma \in S$. (i.e. E_{λ}^{σ} is monic if $|Inv(\lambda)|=0$)

We now show by induction on $|Inv(\lambda)|$ that E_{λ}^{σ} is monic $\forall \lambda \in \mathbb{Z}_{+}^k, \sigma \in S$.

The statement is true for λ s.t. $|Inv(\lambda)|=0$.

Assume the statement is true for λ s.t. $|Inv(\lambda)|=ik$ for some non-negative integer ik .

Consider λ s.t. $|Inv(\lambda)|=ik+1$. ($\therefore Inv(\lambda) \neq \emptyset$)

Take i s.t. $\lambda_{i+1} > \lambda_i$ (equivalently, $(s_i \lambda)_i > (s_i \lambda)_{i+1}$). (i.e. $i \in Inv(\lambda)$)

$$\text{If } s_i \sigma > \sigma, \text{ then } E_{\lambda}^{\sigma} = q^{-1} T_i E_{s_i \lambda}^{s_i \sigma}.$$

$$\text{If } s_i \sigma < \sigma, \text{ then } E_{\lambda}^{\sigma} = q^0 T_i^{-1} E_{s_i \lambda}^{s_i \sigma} = (q^{-1} T_i + (q^{-1}-1)) E_{s_i \lambda}^{s_i \sigma} = q^{-1} T_i E_{s_i \lambda}^{s_i \sigma} + (q^{-1}-1) E_{s_i \lambda}^{s_i \sigma}$$

$$\therefore |Inv(s_i \lambda)| = |Inv(\lambda)| - 1 < |Inv(\lambda)|$$

\therefore By induction assumption, $E_{s_i \lambda}^{s_i \sigma}$ is monic (and triangular by def.).

$$\begin{aligned} \therefore T_i(x^{s_i \lambda} + \sum_{\mu < s_i \lambda} c_{\mu} x^{\mu}) &= q^{\frac{s_i(s_i \lambda)}{2}} + \sum_{\mu < s_i \lambda} d_{\mu} x^{\mu} \quad \text{for some } d_{\mu} \in \mathbb{Q}(q) \\ &= q^{\lambda} + \sum_{\mu < \lambda} d_{\mu} x^{\mu} \end{aligned}$$

$$\therefore q^{-1} T_i E_{s_i \lambda}^{s_i \sigma} = x^{\lambda} + \sum_{\mu < \lambda} q^{-1} d_{\mu} x^{\mu} \quad (\because E_{\lambda}^{\sigma} \text{ is monic if } s_i \sigma > \sigma)$$

$$\text{Note that } E_{s_i \lambda}^{s_i \sigma} = x^{s_i \lambda} + \sum_{\mu < s_i \lambda} c_{\mu} x^{\mu} \text{ and } \lambda > s_i \lambda \text{ when } \lambda_{i+1} > \lambda_i$$

$$\therefore E_{s_i \lambda}^{s_i \sigma} \text{ has no } x^{\lambda} \text{ term}$$

$$\therefore q^{-1} T_i E_{s_i \lambda}^{s_i \sigma} + (q^{-1}-1) E_{s_i \lambda}^{s_i \sigma} = x^{\lambda} + (q^{-1}-1)x^{s_i \lambda} + \sum_{\mu < \lambda} q^{-1} d_{\mu} x^{\mu} + \sum_{\mu < s_i \lambda} c_{\mu} x^{\mu} = x^{\lambda} + \sum_{\mu < \lambda} \tilde{c}_{\mu} x^{\mu} \quad \text{for some } \tilde{c}_{\mu} \in \mathbb{Q}(q)$$

Hence E_{λ}^{σ} is monic if $s_i \sigma < \sigma$.

\therefore The statement is true by induction.

\therefore The twisted non-symmetric polynomials E_{λ}^{σ} are monic and triangular, i.e. has the form $x^{\lambda} + \sum_{\mu < \lambda} c_{\mu} x^{\mu}$ for some $c_{\mu} \in \mathbb{Q}(q)$.

e.g. Compute $E_{s_1 s_2}^{s_1}$

Method I) Construct the non-twisted version

$$S=S_1 \Rightarrow s_1 \sigma = id < \sigma \quad \therefore s_2 s_1 (974) = s_2 (794) = 794$$

$$\therefore E_{794} = q^{-2} T_2 T_1 x^{974}$$

$$\text{Also, } \lambda_1 < \lambda_2 \Rightarrow 1(\lambda_1 \geq \lambda_2) = 0$$

$$\therefore E_{794}^{s_1} = q^0 T_1^{-1} E_{794}^{id} = (q^{-1} T_1 + q^{-1}-1) E_{794} = (q^{-1} T_1 + q^{-1}-1)(q^{-2} T_2 T_1 x^{974}) = q^{-3} T_1 T_2 T_1 x^{974} + (q^{-3} - q^{-2}) T_2 T_1 x^{974}$$

Method II) Make the subscript dominant and use $E_{\lambda}^{\sigma} = x^{\lambda}$

$$s_1 s_2 \cdot 479 = 974,$$

$$\lambda=479: s_1 s_2 = id < s_1, \text{ and } 4 < 7 \Rightarrow 1(\lambda_1 \geq \lambda_2) = 0$$

$$\therefore E_{479}^{s_1} = q^0 T_1^{-1} E_{479}^{id}$$

$$\lambda=794: s_2 id = s_2 > id \text{ and } 4 < 9 \Rightarrow 1(\lambda_2 \leq \lambda_1) = 1$$

$$\therefore E_{794}^{s_1} = T_1^{-1} (q^{-1} T_2 E_{794}^{s_2}) = q^{-1} T_1 T_2 E_{794}^{s_2}$$

$$\lambda=794: s_1 s_2 > s_2 \text{ and } 7 < 9 \Rightarrow 1(\lambda_1 \leq \lambda_2) = 1$$

$$\therefore E_{794}^{s_1} = q^1 T_1 T_2 (q^{-1} T_1 E_{794}^{s_2}) = q^{-2} T_1^{-1} T_2 T_1 x^{974} = q^{-2} (q^1 T_1 + q^{-1}-1) T_2 T_1 x^{974} = (q^{-3} T_1 T_2 T_1 + (q^{-3} - q^{-2}) T_2 T_1) x^{974}$$

$$\begin{aligned} T_1 x^{974} &= (q^{-1}) x^{974} + (q-1)x^{884} + q x^{794}, \quad T_2 T_1 x^{974} = (q-1)^2 (x^{974} + x^{965} + x^{956} + x^{884} + x^{875} + x^{866} + x^{857} + x^{766} + x^{765} + x^{764} + x^{763} + x^{762} + x^{761} + x^{760} + x^{759} + x^{758} + x^{757} + x^{756} + x^{755} + x^{754} + x^{753} + x^{752} + x^{751} + x^{750} + x^{749} + x^{748} + x^{747} + x^{746} + x^{745} + x^{744} + x^{743} + x^{742} + x^{741} + x^{740} + x^{739} + x^{738} + x^{737} + x^{736} + x^{735} + x^{734} + x^{733} + x^{732} + x^{731} + x^{730} + x^{729} + x^{728} + x^{727} + x^{726} + x^{725} + x^{724} + x^{723} + x^{722} + x^{721} + x^{720} + x^{719} + x^{718} + x^{717} + x^{716} + x^{715} + x^{714} + x^{713} + x^{712} + x^{711} + x^{710} + x^{709} + x^{708} + x^{707} + x^{706} + x^{705} + x^{704} + x^{703} + x^{702} + x^{701} + x^{700} + x^{699} + x^{698} + x^{697} + x^{696} + x^{695} + x^{694} + x^{693} + x^{692} + x^{691} + x^{690} + x^{689} + x^{688} + x^{687} + x^{686} + x^{685} + 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+ x^{82} + x^{81} + x^{80} + x^{79} + x^{78} + x^{77} + x^{76} + x^{75} + x^{74} + x^{73} + x^{72} + x^{71} + x^{70} + x^{69} + x^{68} + x^{67} + x^{66} + x^{65} + x^{64} + x^{63} + x^{62} + x^{61} + x^{60} + x^{59} + x^{58} + x^{57} + x^{56} + x^{55} + x^{54} + x^{53} + x^{52} + x^{51} + x^{50} + x^{49} + x^{48} + x^{47} + x^{46} + x^{45} + x^{44} + x^{43} + x^{42} + x^{41} + x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{6} + x^{5} + x^{4} + x^{3} + x^{2} + x^{1} + x^{0} + x^{199} + \dots$$

Define an inner product on $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$:

$$\langle f, g \rangle_q := \langle x^\alpha \rangle f g \prod_{i=1}^n \frac{1 - \frac{x_i}{q}}{1 - q^{-1} \frac{x_i}{q}}.$$

Lemma: T_i are self-adjoint w.r.t. $\langle \cdot, \cdot \rangle_q$.

Proof: Note that $T_i \circ g$ is self-adjoint iff g is self-adjoint.

$$\text{It is because } \langle (T_i \circ g) f, g \rangle_q = \langle T_i f, g \rangle_q - q \langle f, g \rangle_q$$

$$\langle f, (T_i \circ g) g \rangle_q = \langle f, T_i g \rangle_q - q \langle f, g \rangle_q$$

$$\therefore \langle (T_i \circ g) f, g \rangle_q = \langle f, (T_i \circ g) g \rangle_q \iff \langle T_i f, g \rangle_q = \langle f, T_i g \rangle_q \quad \forall f, g \in \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}].$$

Note that

$$T_i \circ g = g(T_{i-1}) + \frac{1 - q}{1 - \frac{x_{i+1}}{x_i}} (S_{i-1}) = \frac{1 - q^{-1} \frac{x_{i+1}}{x_i}}{1 - \frac{x_{i+1}}{x_i}} (S_{i-1}) = \frac{q}{1 - \frac{x_{i+1}}{x_i}} (S_{i-1})$$

$$\begin{aligned} \therefore \langle (T_i \circ g) f, g \rangle_q &= \langle g \frac{1 - q^{-1} \frac{x_{i+1}}{x_i}}{1 - \frac{x_{i+1}}{x_i}} (S_{i-1}) f, g \rangle_q = \langle x^\alpha \rangle \left(g \frac{1 - q^{-1} \frac{x_{i+1}}{x_i}}{1 - \frac{x_{i+1}}{x_i}} (S_{i-1}) f g \prod_{\substack{1 \leq j \leq k \\ (j,k) \neq (i,i+1)}} \frac{1 - \frac{x_j}{q}}{1 - q^{-1} \frac{x_j}{q}} \right) = \langle x^\alpha \rangle [S_i f] g - fg \prod_{\substack{1 \leq j \leq k \\ (j,k) \neq (i,i+1)}} \frac{1 - \frac{x_j}{q}}{1 - q^{-1} \frac{x_j}{q}} \\ \langle f, (T_i \circ g) g \rangle_q &= \langle f, g \frac{1 - q^{-1} \frac{x_{i+1}}{x_i}}{1 - \frac{x_{i+1}}{x_i}} (S_{i-1}) g \rangle_q = \langle x^\alpha \rangle \left(g \frac{1 - q^{-1} \frac{x_{i+1}}{x_i}}{1 - \frac{x_{i+1}}{x_i}} (S_{i-1}) g f g \prod_{\substack{1 \leq j \leq k \\ (j,k) \neq (i,i+1)}} \frac{1 - \frac{x_j}{q}}{1 - q^{-1} \frac{x_j}{q}} \right) = \langle x^\alpha \rangle [f(S_i g)] - fg \prod_{\substack{1 \leq j \leq k \\ (j,k) \neq (i,i+1)}} \frac{1 - \frac{x_j}{q}}{1 - q^{-1} \frac{x_j}{q}} \end{aligned}$$

Hence it suffices to show

$$\langle x^\alpha \rangle (S_i f) g \prod_{\substack{1 \leq j \leq k \\ (j,k) \neq (i,i+1)}} \frac{1 - \frac{x_j}{q}}{1 - q^{-1} \frac{x_j}{q}} = \langle x^\alpha \rangle f (S_i g) \prod_{\substack{1 \leq j \leq k \\ (j,k) \neq (i,i+1)}} \frac{1 - \frac{x_j}{q}}{1 - q^{-1} \frac{x_j}{q}}$$

$$\text{Note that } S_i \left(\prod_{\substack{1 \leq j \leq k \\ (j,k) \neq (i,i+1)}} \frac{1 - \frac{x_j}{q}}{1 - q^{-1} \frac{x_j}{q}} \right) = \prod_{\substack{1 \leq j \leq k \\ (j,k) \neq (i,i+1)}} \frac{1 - \frac{x_j}{q}}{1 - q^{-1} \frac{x_j}{q}} \quad \text{and } \langle x^\alpha \rangle (\phi(x_1, \dots, x_k)) = \langle x^\alpha \rangle (S_i \phi(x_1, \dots, x_k))$$

$$\therefore \langle x^\alpha \rangle (S_i f) g \prod_{\substack{1 \leq j \leq k \\ (j,k) \neq (i,i+1)}} \frac{1 - \frac{x_j}{q}}{1 - q^{-1} \frac{x_j}{q}} = \langle x^\alpha \rangle S_i \left((S_i f) g \prod_{\substack{1 \leq j \leq k \\ (j,k) \neq (i,i+1)}} \frac{1 - \frac{x_j}{q}}{1 - q^{-1} \frac{x_j}{q}} \right) = \langle x^\alpha \rangle f (S_i g) \prod_{\substack{1 \leq j \leq k \\ (j,k) \neq (i,i+1)}} \frac{1 - \frac{x_j}{q}}{1 - q^{-1} \frac{x_j}{q}} \quad \text{and result follows.}$$

□

Prop: For any $\tau \in S_n$, $\{E_\lambda^\tau\}_{\lambda \in \mathbb{Z}^n}$ and $\{\bar{F}_\lambda^\tau\}_{\lambda \in \mathbb{Z}^n}$ are dual bases of $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ w.r.t. $\langle \cdot, \cdot \rangle_q$.

$$\text{i.e. } \langle E_\lambda^\tau, \bar{F}_\mu^\tau \rangle_q = \delta_{\lambda\mu} \quad \forall \lambda, \mu \in \mathbb{Z}^n, \tau \in S_n.$$

Proof: It is equivalent to show $\langle E_\lambda^\tau, E_{\tau(\mu)}^{\sigma\omega} \rangle_q = \delta_{\lambda\mu}$. (Note: They are basis b/c E_λ^τ has a monic triangular form and F_λ^τ are "basically" E_λ^τ 's with $x_i \mapsto x_i^{-1}$)
 $\frac{q}{q-1} \delta_{\lambda(\tau(\mu))} = \delta_{\lambda\mu}$

$$\langle E_\lambda^\tau, E_{\tau(\mu)}^{\sigma\omega} \rangle_q = \begin{cases} q & \text{if } \lambda(\tau(\mu)) = \lambda \text{ and } \sigma\omega > \tau\omega \\ 0 & \text{if } \lambda(\tau(\mu)) \neq \lambda \text{ or } \sigma\omega \leq \tau\omega \end{cases}$$

Since T_i is self-adjoint w.r.t. $\langle \cdot, \cdot \rangle_q$, we have

$$\langle E_\lambda^\tau, E_{\tau(\mu)}^{\sigma\omega} \rangle_q = \begin{cases} q & \text{if } \lambda(\tau(\mu)) = \lambda \text{ and } \sigma\omega > \tau\omega \\ 0 & \text{if } \lambda(\tau(\mu)) \neq \lambda \text{ or } \sigma\omega \leq \tau\omega \end{cases} \quad (\star)$$

Note that if $\lambda = \mu$, then $-\mathbb{1}(\lambda_i \leq \lambda_{i+1}) + \mathbb{1}(\mu_i \leq \mu_{i+1}) = \mathbb{1}(\lambda_i > \lambda_{i+1}) - \mathbb{1}(\mu_i > \mu_{i+1}) = 0$

$$\langle E_\lambda^\tau, E_{\tau(\mu)}^{\sigma\omega} \rangle_q = \begin{cases} \langle E_{\tau(\lambda)}^{\sigma\omega}, E_{\tau(\mu)}^{\sigma\omega} \rangle_q & \text{if } \sigma\omega > \tau\omega \\ 0 & \text{if } \sigma\omega \leq \tau\omega \end{cases}$$

Repeat the recurrence (\star) , we get

$$\langle E_\lambda^\tau, E_{\tau(\mu)}^{\sigma\omega} \rangle_q = q^e \langle E_{\tau(\lambda)}^{\sigma\omega}, E_{\tau(\mu)}^{\sigma\omega} \rangle_q \quad \text{for some } e \in \mathbb{Z} \text{ and } e=0 \text{ for } \lambda=\mu$$

$$-v(\lambda)$$

Take $v \in S_0$ s.t. $v(\mu) = \mu_-$. Then $E_{-\nu(v)}^{\text{vivo}} = E_{\mu}^{\text{vivo}} = x^{\mu_-}$

$$\langle E_{\lambda}^{\sigma}, E_{\mu}^{\text{vivo}} \rangle_q = q^e \langle E_{\nu(v)}^{\sigma}, x^{\mu_-} \rangle_q = q^e \langle x^{\sigma} \rangle E_{\nu(v)}^{\text{vivo}} \prod_{1 \leq j < k \leq l} \frac{1 - \frac{x_k}{x_j}}{1 - \frac{q^{-1}x_k}{q^{-1}x_j}} = q^e \langle x^{\mu_-} \rangle E_{\nu(v)}^{\text{vivo}} \prod_{1 \leq j < k \leq l} \frac{1 - \frac{x_k}{x_j}}{1 - \frac{q^{-1}x_k}{q^{-1}x_j}}$$

Notation: $\text{supp}(f) = \{ \sigma \in \mathbb{Z}_2 : \text{coeff of } x^\sigma \text{ in } f \neq 0 \}$

$$\therefore \prod_{1 \leq j < k \leq l} \frac{1 - \frac{x_k}{x_j}}{1 - \frac{q^{-1}x_k}{q^{-1}x_j}} = \prod_{1 \leq j < k \leq l} \left(1 - \frac{x_k}{x_j} \right) \left(1 + \frac{q^{-1}}{q^{-1}x_k} + \frac{q^2}{q^{-1}x_k^2} + \dots \right)$$

$$= \prod_{1 \leq j < k \leq l} \left[1 + \left(q^{-1} - 1 \right) \frac{x_k}{x_j} + \left(q^{-2} - q^{-1} \right) \frac{x_k^2}{x_j^2} + \left(q^{-3} - q^{-2} \right) \frac{x_k^3}{x_j^3} + \dots \right]$$

$$= \prod_{1 \leq j < k \leq l} \left[1 + q^j \left((1-q) \frac{x_k}{x_j} + q^{-2}(-q) \frac{x_k^2}{x_j^2} + q^{-3}(-q) \frac{x_k^3}{x_j^3} + \dots \right) \right]$$

$$= 1 + \sum_{\alpha \in Q_+ \setminus \{0\}} c_\alpha(q) x^\alpha \quad \text{where } c_\alpha(q) \neq 0 \text{ and } c_\alpha(q) \in \mathbb{Q}(q) \quad \text{is divisible by } 1-q$$

$$\therefore \text{supp} \left(\prod_{1 \leq j < k \leq l} \frac{1 - \frac{x_k}{x_j}}{1 - \frac{q^{-1}x_k}{q^{-1}x_j}} \right) = Q_+$$

We know $\text{supp}(E_{\nu(v)}^{\text{vivo}}) \subseteq \text{conv}(S_0 \cdot \lambda)$, hence if $\langle E_{\lambda}^{\sigma}, E_{\mu}^{\text{vivo}} \rangle_q \neq 0$, i.e. $q^e \langle x^{\mu_-} \rangle E_{\nu(v)}^{\text{vivo}} \prod_{1 \leq j < k \leq l} \frac{1 - \frac{x_k}{x_j}}{1 - \frac{q^{-1}x_k}{q^{-1}x_j}} \neq 0$, then

$$\mu_- \in \text{supp}(E_{\nu(v)}^{\text{vivo}} \cdot \prod_{1 \leq j < k \leq l} \frac{1 - \frac{x_k}{x_j}}{1 - \frac{q^{-1}x_k}{q^{-1}x_j}}).$$

$$\therefore E_{\nu(v)}^{\text{vivo}} \prod_{1 \leq j < k \leq l} \frac{1 - \frac{x_k}{x_j}}{1 - \frac{q^{-1}x_k}{q^{-1}x_j}} = \sum_{\gamma \in \text{conv}(S_0 \cdot \lambda)} d_\gamma x^\gamma \cdot \sum_{\alpha \in Q_+} c_\alpha x^\alpha \quad \text{where } c_0 = 1, c_\alpha, d_\gamma \in \mathbb{Q}(q)$$

$$= \sum_{\substack{\gamma \in \text{conv}(S_0 \cdot \lambda) \\ \alpha \in Q_+}} d_\gamma c_\alpha x^{\gamma+\alpha}$$

$\therefore \mu_- = \gamma + \alpha$ for some $\gamma \in \text{conv}(S_0 \cdot \lambda)$ and $\alpha \in Q_+$ s.t. $d_\gamma c_\alpha \neq 0$

i.e. $\mu_- - \alpha = \gamma \in \text{conv}(S_0 \cdot \lambda)$

$\therefore (\mu_- - Q) \cap \text{conv}(S_0 \cdot \lambda) \neq \emptyset$

$\therefore \gamma - \lambda_- \in Q_+$ for $\gamma \in \text{conv}(S_0 \cdot \lambda)$

$$\therefore \mu_- - \lambda_- = \mu_- - \gamma + \gamma - \lambda_- = \alpha + (\gamma - \lambda_-) \in Q_+$$

$$\therefore w_0(Q_+) = -Q_+$$

$$\therefore \mu_+ + \lambda_+ \in -Q_+ \Rightarrow \lambda_+ - \mu_+ \in Q_+ \Rightarrow \lambda_+ \geq \mu_+$$

$$\therefore \langle E_{\lambda}^{\sigma}, E_{\mu}^{\text{vivo}} \rangle_q \neq 0 \Rightarrow \lambda_+ \geq \mu_+$$

$$\text{Also, } \langle E_{\mu}^{\text{vivo}}, E_{\lambda}^{\sigma} \rangle_q = \langle E_{-\mu}^{\text{vivo}}, E_{-\lambda}^{\sigma} \rangle_q$$

$$\therefore \langle E_{-\mu}^{\text{vivo}}, E_{\lambda}^{\sigma} \rangle_q \neq 0 \Rightarrow (-\mu)_+ \geq (-\lambda)_+ \Leftrightarrow -\mu_+ \geq -\lambda_+ \Leftrightarrow \lambda_+ \geq \mu_+ \Leftrightarrow \lambda_+ \geq \lambda_-$$

$$\therefore \langle E_{\mu}^{\text{vivo}}, E_{\lambda}^{\sigma} \rangle_q = \langle E_{-\mu}^{\text{vivo}}, E_{-\lambda}^{\sigma} \rangle_q$$

$$\therefore \langle E_{\lambda}^{\sigma}, E_{\mu}^{\text{vivo}} \rangle_q \neq 0 \Rightarrow \lambda_+ = \mu_+ \text{ and hence } S_0 \cdot \lambda = S_0 \cdot \mu$$

\therefore We only need to consider $\lambda \in \mathbb{M}$ st. $S_{\lambda} \cdot \lambda = S_{\lambda} \cdot \mu$.

In this case, $(\mu - Q_+) \cap \text{conv}(S_{\lambda} \cdot \lambda) = \emptyset$ b/c $\mu - \alpha \notin \text{conv}(S_{\lambda} \cdot \lambda)$ $\forall \alpha \in Q_+$ not

If $\lambda \neq \mu$, then $v(\lambda) \neq v(\mu)$ i.e. $v(\lambda) > v(\mu)$.

$$\therefore E_{\lambda}^{(v)} = x^{v(\lambda)} + \sum_{\alpha \in Q_+} c_{\alpha} x^{\alpha} \quad \text{and} \quad \mu = \lambda > v(\lambda)$$

$\therefore \mu \notin \text{supp}(E_{\lambda}^{(v)}) \subseteq \text{conv}(S_{\lambda} \cdot \lambda)$

Hence $(\mu - Q_+) \cap \text{supp}(E_{\lambda}^{(v)}) = \emptyset$. Recall

$$E_{\lambda}^{(v)} \prod_{1 \leq j \leq k \leq l} \frac{1 - \frac{x_j}{x_k}}{1 - q \frac{x_j}{x_k}} = \sum_{\substack{\gamma \in \text{Conv}(S_{\lambda} \cdot \lambda) \\ \alpha \in Q_+}} d_{\gamma, \alpha} c_{\alpha} x^{\gamma + \alpha}$$

$\therefore \mu \in \text{supp}\left(E_{\lambda}^{(v)} \prod_{1 \leq j \leq k \leq l} \frac{1 - \frac{x_j}{x_k}}{1 - q \frac{x_j}{x_k}}\right) \Rightarrow \mu = \gamma + \alpha \text{ for some } \gamma \in \text{Conv}(S_{\lambda} \cdot \lambda) \text{ and } \alpha \in Q_+ \text{ st. } d_{\gamma, \alpha} c_{\alpha} \neq 0$

i.e. $\mu - \alpha = \gamma \in \text{supp}(E_{\lambda}^{(v)}) \Rightarrow (\mu - Q_+) \cap \text{supp}(E_{\lambda}^{(v)}) \neq \emptyset$ if $\mu \in \text{supp}\left(E_{\lambda}^{(v)} \prod_{1 \leq j \leq k \leq l} \frac{1 - \frac{x_j}{x_k}}{1 - q \frac{x_j}{x_k}}\right)$.
 b/c $d_{\gamma, \alpha} \neq 0$

$$\therefore (\mu - Q_+) \cap \text{supp}(E_{\lambda}^{(v)}) = \emptyset \Rightarrow \mu \notin \text{supp}\left(E_{\lambda}^{(v)} \prod_{1 \leq j \leq k \leq l} \frac{1 - \frac{x_j}{x_k}}{1 - q \frac{x_j}{x_k}}\right) \Rightarrow \langle E_{\lambda}^{(v)}, E_{-\mu}^{(v)} \rangle_{\frac{1}{q}} = 0$$

Hence $\lambda \neq \mu \Rightarrow \langle E_{\lambda}^{(v)}, E_{-\mu}^{(v)} \rangle_{\frac{1}{q}} = 0$

$$\text{If } \lambda = \mu, \text{ then } \langle E_{\lambda}^{(v)}, E_{-\lambda}^{(v)} \rangle_{\frac{1}{q}} = \langle x^{\lambda} \rangle_{E_{\lambda}^{(v)}} \prod_{1 \leq j \leq k \leq l} \frac{1 - \frac{x_j}{x_k}}{1 - q \frac{x_j}{x_k}} = \langle x^{\lambda} \rangle_{E_{\lambda}^{(v)}} \prod_{1 \leq j \leq k \leq l} \frac{1 - \frac{x_j}{x_k}}{1 - q \frac{x_j}{x_k}}$$

$$\therefore \text{supp}(E_{\lambda}^{(v)}) = \lambda + Q_+ \text{ and } \text{supp}\left(\prod_{1 \leq j \leq k \leq l} \frac{1 - \frac{x_j}{x_k}}{1 - q \frac{x_j}{x_k}}\right) = Q_+$$

\therefore The only way to form x^{λ} in $E_{\lambda}^{(v)}$, $\prod_{1 \leq j \leq k \leq l} \frac{1 - \frac{x_j}{x_k}}{1 - q \frac{x_j}{x_k}}$ is taking the product of the x^{λ} term in $E_{\lambda}^{(v)}$ and the constant term in $\prod_{1 \leq j \leq k \leq l} \frac{1 - \frac{x_j}{x_k}}{1 - q \frac{x_j}{x_k}}$.

i.e. coeff. of x^{λ} in $E_{\lambda}^{(v)}$, $\prod_{1 \leq j \leq k \leq l} \frac{1 - \frac{x_j}{x_k}}{1 - q \frac{x_j}{x_k}}$ is the product of the coeff. of x^{λ} in $E_{\lambda}^{(v)}$ and the constant term in $\prod_{1 \leq j \leq k \leq l} \frac{1 - \frac{x_j}{x_k}}{1 - q \frac{x_j}{x_k}}$.

$$\therefore E_{\lambda}^{(v)} = x^{\lambda} + \sum_{\alpha \in Q_+} c_{\alpha} x^{\alpha} \text{ for some } c_{\alpha} \in \mathbb{Q}(q) \text{ and } \prod_{1 \leq j \leq k \leq l} \frac{1 - \frac{x_j}{x_k}}{1 - q \frac{x_j}{x_k}} = 1 + \sum_{\alpha \in Q_+ \setminus \{\lambda\}} c_{\alpha} q^{\alpha} x^{\alpha} \text{ whose constant term is 1}$$

$$\therefore \text{coeff. of } x^{\lambda} \text{ in } E_{\lambda}^{(v)} \cdot \prod_{1 \leq j \leq k \leq l} \frac{1 - \frac{x_j}{x_k}}{1 - q \frac{x_j}{x_k}} = 1 \times 1 = 1$$

$$\text{i.e. } \langle E_{\lambda}^{(v)}, E_{-\lambda}^{(v)} \rangle_{\frac{1}{q}} = \langle x^{\lambda} \rangle_{E_{\lambda}^{(v)}} \prod_{1 \leq j \leq k \leq l} \frac{1 - \frac{x_j}{x_k}}{1 - q \frac{x_j}{x_k}} = 1.$$

As a result, $\langle E_{\lambda}^{(v)}, E_{-\mu}^{(v)} \rangle_{\frac{1}{q}} = \delta_{\lambda, \mu}$. □

Lemma: Let $\lambda \in \mathbb{I}^l$. Suppose $\exists k \in \mathbb{I}$ st. $\lambda_i \geq \lambda_j \forall i \leq k$ and $j > k$.

Given $\sigma \in S_{\lambda}$, define $\sigma_i \in S_{\lambda}$ and $\tau_i \in S_{\lambda-k}$ st. $\sigma_i(1), \dots, \sigma_i(k)$ are in the same relative order as $\sigma(1), \dots, \sigma(k)$, and $\tau_i(1), \dots, \tau_i(k)$ are in the same relative order as $\sigma(k+1), \dots, \sigma(l)$. Then

$$E_{\lambda}^{(\sigma)}(x_1, \dots, x_l; q) = E_{\sigma(1), \dots, \sigma(k)}^{(\sigma_1)}(x_1, \dots, x_k; q) E_{\sigma(k+1), \dots, \sigma(l)}^{(\sigma_2)}(x_{k+1}, \dots, x_l; q).$$

Proof: If $\lambda = \lambda_+$, then $E_{\lambda}^{(\sigma)} = x^{\lambda} = (x_1, \dots, x_k) \cdot (x_{k+1}, \dots, x_l) = E_{\sigma(1), \dots, \sigma(k)}^{(\sigma_1)}(x_1, \dots, x_k; q) E_{\sigma(k+1), \dots, \sigma(l)}^{(\sigma_2)}(x_{k+1}, \dots, x_l; q)$. Hence the statement is trivial for $\lambda = \lambda_+$.

Otherwise, $E_{\lambda}^{(\sigma)}$ can be constructed by the recurrence

$$E_{\lambda}^{(\sigma)} = \begin{cases} 0 & \text{if } \lambda_i > \lambda_{i+1} \\ \sum_{j=1}^l \langle \lambda_i - \lambda_j, \tau_j \rangle_{\frac{1}{q}} E_{\sigma(1), \dots, \sigma(k)}^{(\sigma_1)}(x_1, \dots, x_k; q) E_{\sigma(k+1), \dots, \sigma(l)}^{(\sigma_2)}(x_{k+1}, \dots, x_l; q) & \text{if } \lambda_i < \lambda_{i+1} \end{cases}$$

k can be any index in this case.

on induction on $|\text{Inv}(\lambda)| = |\{(a, b) : a < b \text{ and } \lambda_a < \lambda_b\}|$

Hence we only need to consider $1 \leq a < b \leq k$ and $k < a < b \leq l$.

\therefore We only take i st. $s_i \in S_k \times S_{k+1} \subseteq S_\ell$ when using the recurrence.

\because Both $E_{(x_1, \dots, x_{k+1})}^{\sigma^{-1}}(x_1, \dots, x_{k+1}; q)$ and $E_{(x_1, \dots, x_{k+1})}^{\sigma^*}(x_1, \dots, x_{k+1}; q)$ satisfy the same recurrence and the recurrence determines $E_{\lambda}^{\sigma^{-1}}$, result follows. \square

e.g. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 7 & 1 & 4 & 8 & 3 & 2 & 6 \end{pmatrix}$, $\lambda = (6, 9, 5, 6, 7, 1, 3, 2)$. Hence $k = 5$.

$$\therefore \sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Hence $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 6 & 4 & 1 & 8 & 2 & 5 \end{pmatrix}$, $\sigma_1^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}$, $\sigma_2^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. By the lemma above, we have

$$E_{69567132}^{\sigma^{-1}}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8; q) = E_{69567}^{\sigma_1^{-1}}(x_4, x_5, x_3, x_4, x_5; q) E_{132}^{\sigma_2^{-1}}(x_6, x_7, x_8; q).$$