Def: A S.E. lattice path γ from $(0,n)$ to $(m,0)$ $(m,n\in\mathbb{Z}^+)$ is admissible if it starts with a S-step and ends with an E-step.

Notation:

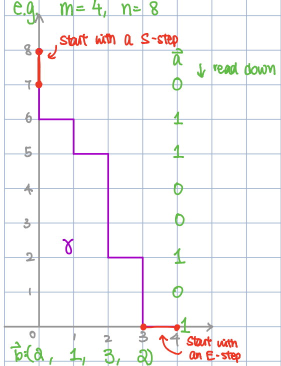
- $\vec{b}(\gamma)=(b_1,b_2,\dots,b_m)$ where $b_i=\#$ S-steps on $x=i-1$ $\forall\ 1\le i\le m$ $(\therefore b_1\geq 1)$
- $\vec{a}(\gamma)=(a_n,a_{n-1},\dots,a_1)$ where $a_j=\#$ E-steps on $y=j-1$ $\forall\ 1\le j\le n$ $(\therefore a_1\geq 1)$
- $D_\gamma:=D_{\vec{b}(\gamma)}$, $E_\gamma:=E_{\vec{a}(\gamma)}$.
- γ^* = transpose of γ (S-step $\leftrightarrow$ E-step) and hence also admissible

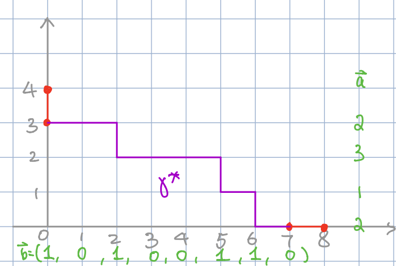
$\therefore$ $\vec{b}(\gamma^*)=(a_1,\dots,a_n)$

$\vec{a}(\gamma^*)=(b_m,b_{m-1},\dots,b_1)$

$E_\gamma=\Phi(D_{\gamma^*})$ (Recall $E_{r_\ell r_{\ell-1}\cdots r_1}:=\Phi(D_{r_1r_2\cdots r_\ell})$ $\forall (r_1,\dots,r_\ell)\in\mathbb{Z}^\ell$)

e.g. $m=4,\ n=8$





γ is an admissible path from $(0,8)$ to $(4,0)$	γ^* is an admissible path from $(0,4)$ to $(8,0)$.
$\vec{b}(\gamma)=(2,1,3,2)$	$\vec{b}(\gamma^*)=(1,0,1,0,0,1,1,0)=\mathrm{rev}(\vec{a}(\gamma))$
$\vec{a}(\gamma)=(0,1,1,0,0,1,0,1)$	$\vec{a}(\gamma^*)=(2,3,1,2)=\mathrm{rev}(\vec{b}(\gamma))$
$D_\gamma=D_{2132}$	$D_{\gamma^*}=D_{10100110}$
$E_\gamma=E_{01100101}=\Phi(D_{10100110})$	$E_{\gamma^*}=E_{2312}=\Phi(D_{2132})=\Phi(D_\gamma)$

Prop 4.3.3: If γ is an admissible path, then $D_\gamma=E_\gamma$.

Proof: Let γ be an admissible path from $(0,n)$ to $(m,0)$, $m,n\in\mathbb{Z}^+$.

When $n=1$:

Recall: $\mathrm{Ad}_{p_1}(x^{1,0})\,p_1[-Mx^{r,1}]=-M\left(\omega p_1(z)\big|_{z^k\mapsto p_1[x^{r+k,1}]}\right)=-Mz\big|_{z\mapsto p_1[x^{r+1,1}]}=-Mp_1[x^{r+1,1}]=p_1[-Mx^{r+1,1}]$

$\therefore\ \mathrm{Ad}_{p_1}(x^{1,0})\,p_1[-Mx^{1,1}]=p_1[-Mx^{2,1}]$

$(\mathrm{Ad}_{p_1}(x^{1,0}))^r\,p_1[-Mx^{1,1}]=(\mathrm{Ad}_{p_1}(x^{1,0}))^{r-1}\,p_1[-Mx^{2,1}]=(\mathrm{Ad}_{p_1}(x^{1,0}))^{r-2}\,p_1[-Mx^{3,1}]=\cdots=p_1[-Mx^{r+1,1}]$

$D_\gamma=D_{1,\underbrace{0,0,\dots,0}_{m-1}}$, $E_\gamma=E_m=p_1[-Mx^{m,1}]$

When $m=1$, $D_\gamma=D_1=p_1[-Mx^{1,1}]=E_1$

set $r=m-1$

When $m>1$, $E_\gamma=p_1[-Mx^{m,1}]=(\mathrm{Ad}_{p_1}(x^{1,0}))^{m-1}\,p_1[-Mx^{1,1}]=(\mathrm{Ad}_{p_1}(x^{1,0}))^{m-1}D_1$

Recall for any algebra A containing a copy of Λ, $(\mathrm{Ad}_{p_n})\xi=[p_n,\xi]$ $\forall \xi\in A$

$\therefore\ (\mathrm{Ad}_{p_1}(x^{1,0}))D_{10^k}=[p_1(x^{1,0}),D_{10^k}]=[-\frac{1}{M}\underbrace{p_1[-Mx^{1,0}]}_{D_0},D_{10^k}]=-\frac{1}{M}[D_0,D_{10^k}]$

Recall Prop 4.2.1 (i):

$$[D_a,D_{b_1,\dots,b_\ell}]=-M\sum_{i=1}^{\ell}\sum_{j=a+1}^{b_i}{}^{\#}\,D_{b_1,\dots,b_{i-1},j,a+b_i-j,b_{i+1},\dots,b_\ell}$$

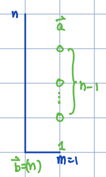
$a=0$

$$\therefore\ [D_0,D_{1,\underbrace{0,\dots,0}_{k}}]=-M\sum_{i=1}^{k+1}\sum_{j=1}^{b_i}{}^{\#}\,D_{b_1,\dots,b_{i-1},j,b_i-j,b_{i+1},\dots,b_{k+1}}=-MD_{\underset{j}{1},\overset{b_1-j}{1-1},\underbrace{0,0,\dots,0}_{b_2,\dots,b_{k+1}\ \therefore\ k\ \text{'0's}}}=-MD_{1,\underbrace{0,\dots,0}_{k+1}}\Rightarrow -\frac{1}{M}[D_0,D_{10^k}]=D_{1\underbrace{0\cdots0}_{k+1}}$$

$\sum_{j=1}^{0}{}^{\#}$ = empty sum, $\sum_{j=1}^{1}{}^{\#}=\sum_{j=1}^{1}$ (Hence only consider $i=1$ case)

$\therefore\ (\mathrm{Ad}_{p_1}(X^{1,0}))D_{10^k}=D_{10^{k+1}}\Rightarrow (\mathrm{Ad}_{p_1}(x^{1,0}))^{m-1}D_1=(\mathrm{Ad}_{p_1}(x^{1,0}))^{m-2}D_{10}=\cdots=D_{10^{m-1}}\Rightarrow E_\gamma=E_m=D_{10^{m-1}}=D_\gamma$ for $n=1$ and $m\in\mathbb{Z}^+$.

When $m=1$,



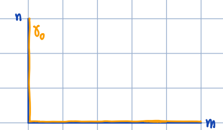
Recall $E_n = \Phi D_n$ and $\Phi^2 = id$

$$D_\gamma = D_n = \Phi E_n = \Phi D_{1,0^{n-1}} = E_{0^{n-1},1} = E_\gamma \quad \forall n \in \mathbb{Z}^+$$

When $m, n > 1$, we proceed by induction.

Assume $D_\gamma = E_\gamma$ for all admissible paths γ from $(0,n')$ to $(m',0)$ when $m' \leq m$ and $n' \leq n$ and $(m',n') \neq (m,n)$.

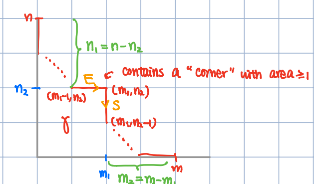
Denote γ_0: path with 0 area :



Suppose $\gamma \neq \gamma_0$. Then γ contains an E-step from (m_1-1, n_2) to (m_1, n_2) and a S-step (m_1, n_2) to (m_1, n_2-1) where $1 \leq m_1 < m$, $1 \leq n_2 < n$. Set $\underbrace{m_2 = m - m_1}$, $\underbrace{n_1 = n - n_2}$

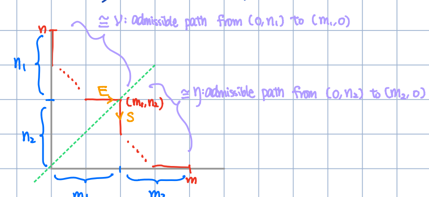
$m_1 + m_2 = m$, $n_1 + n_2 = n$

$1 \leq m_1, m_2 < m$, $1 \leq n_1, n_2 < n$



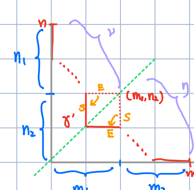
Then $\gamma = \nu \cdot \eta$ for shorter admissible paths ν and η where $\nu \cdot \eta$ is defined to be the lattice path obtained by coinciding end of ν with start of η.

Here ν is an admissible path from $(0,n_1)$ to $(m_1,0)$ and η is an admissible path from $(0,n_2)$ to $(m_2,0)$, and we form γ by putting $(m_1,0)$ & $(0,n_2)$ at (m_1,n_2).



Define $\nu \cdot' \eta$ to be the path obtained by first replacing the last E-step of ν to a S-step and the first S-step of η to an E-step and coincide the new end of ν and new start of η together.

Denote $\gamma' = \nu \cdot' \eta$.



Recall $D_{b_1 b_2 \cdots b_l} D_{b_{l+1}, \cdots, b_n} = D_{b_1 \cdots b_n} - qt\, D_{b_1 \cdots b_{l-1}\, b_l+1\, b_{l+1}-1\, b_{l+2} \cdots b_n}$ (cf. §4.1)

$E_{a_n a_{n-1} \cdots a_{l+1}} E_{a_l \cdots a_1} = E_{a_n \cdots a_1} - qt\, E_{a_n \cdots a_{l-1}\, a_{l+1}-1\, a_l+1\, a_{l-1} \cdots a_1}$

$$\therefore D_\nu D_\eta = D_\gamma - qt\, D_{\gamma'}$$

$$E_\nu E_\eta = E_\gamma - qt\, E_{\gamma'}$$

By induction, $D_\nu = E_\nu$, $D_\eta = E_\eta$. Hence $D_\gamma - qt\, D_{\gamma'} = E_\gamma - qt\, E_{\gamma'} \Rightarrow D_\gamma - E_\gamma = qt\,(D_{\gamma'} - E_{\gamma'}) = (qt)^2 (D_{\gamma''} - E_{\gamma''}) = \cdots = (qt)^{\text{area}(\gamma)} (D_{\gamma_0} - E_{\gamma_0})$

Hence it suffices to prove $D_{\gamma_0} = E_{\gamma_0}$. i.e. $D_{n,\underbrace{0,0,\cdots,0}_{m-1}} = E_{\underbrace{0,0,\cdots,0}_{n-1},m}$

We can assume $D_{n,\underbrace{0,0,\ldots,0}_{m-a}} = E_{\underbrace{0,0,\ldots,0}_{n-1},m-1}$ because the corresponding admissible path starts from $(0,n)$ and ends at $(0,m-1)$. $\swarrow < m$

$$[D_0, D_{n,\underbrace{0,\ldots,0}_{m-a}}] = [\rho_1[-MX^{1,0}], D_{n,\underbrace{0,\ldots,0}_{m-a}}] = -M[\rho_1(X^{1,0}), D_{n,\underbrace{0\ldots0}_{m-2}}] = -M(\mathrm{Ad}\,\rho_1(X^{1,0}))D_{n,\underbrace{0,\ldots,0}_{m-2}} = -M(\mathrm{Ad}\,\rho_1(X^{1,0}))E_{\underbrace{0,\ldots,0}_{n-1},m-1}$$

Recall Prop 4.2.1 (i):

$$[D_{\underset{\uparrow 0}{a}}, D_{b_1,\ldots,b_\ell}] = -M\sum_{i=1}^{\ell}\sum_{j=a+1}^{b_i}{}^{\#} D_{b_1,\ldots,b_{i-1},j,\overset{0\swarrow}{a}+b_i-j,b_{i+1},\ldots,b_\ell}$$

$$\therefore\ [D_0, D_{n,\underbrace{0,\ldots,0}_{m-2}}] = -M\sum_{i=1}^{m-1}\sum_{j=1}^{b_i}{}^{\#} D_{b_1,\ldots,b_{i-1},j,b_i-j,b_{i+1},\ldots,b_{k+1}} = -M\sum_{j=1}^{n} D_{j,n-j,\underbrace{0,0,\ldots,0}_{m-2}}$$

$\sum_{j=1}^{0}{}^{\#}$ = empty sum, $\sum_{j=1}^{n}{}^{\#} = \sum_{j=1}^{n}$ (Hence only consider $i=1$ case)

Recall Lemma 4.1.2:

$$(\mathrm{Ad}\,f(X^{1,0}))E_{a_n,\ldots,a_1} = \psi^{op}\left(\frac{\omega f(z_1,\ldots,z_n)z_1^{a_1}\cdots z_n^{a_n}}{\prod_{i=1}^{n-1}(1-qt\frac{z_i}{z_{i+1}})}\right)$$

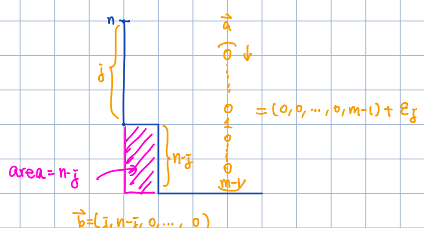
$$\therefore\ (\mathrm{Ad}\,\rho_1(X^{1,0}))E_{\underbrace{0,0,\ldots,0}_{n-1},m-1} = \psi^{op}\left(\frac{\rho_1(z_1,\ldots,z_n)z_1^{m-1}\cdots z_{n-1}^{0}z_n^{0}}{\prod_{i=1}^{n-1}(1-qt\frac{z_i}{z_{i+1}})}\right) = \psi^{op}\left(\frac{z_1^m+z_1^{m-1}z_2+z_1^{m-1}z_3+\cdots+z_1^{m-1}z_n}{\prod_{i=1}^{n-1}(1-qt\frac{z_i}{z_{i+1}})}\right) = \sum_{j=1}^{n}\psi^{op}\left(\frac{z_1^{m-1}z_j}{\prod_{i=1}^{n-1}(1-qt\frac{z_i}{z_{i+1}})}\right) = \sum_{j=1}^{n} E_{(\underbrace{0,0,\ldots,0}_{n-1},m-1)+\varepsilon_j}$$

$(0,0,\ldots,0,1,0,\ldots,0)$ $\uparrow$ j^{th} position

Hence $\cancel{-M}\sum_{j=1}^{n} D_{j,n-j,\underbrace{0,\ldots,0}_{m-2}} = \cancel{-M}\sum_{j=1}^{n} E_{(\underbrace{0,0,\ldots,0}_{n-1},m-1)+\varepsilon_j}$

$$\underset{(\text{when } j=n)}{D_{\gamma_0}} + \sum_{j=1}^{n-1} D_{j,n-j,\underbrace{0,\ldots,0}_{m-2}} = \underset{\text{when } j=n}{E_{\gamma_0}} + \sum_{j=1}^{n-1} E_{(\underbrace{0,\ldots,0}_{n-1},m-1)+\varepsilon_j} \Rightarrow D_{\gamma_0} - E_{\gamma_0} + \sum_{j=1}^{n-1}\left(D_{j,n-j,\underbrace{0,\ldots,0}_{m-2}} - E_{(\underbrace{0,\ldots,0}_{n-1},m-1)+\varepsilon_j}\right) = 0$$

For $1 \le j \le n-1$:



Corresponding D and E:

$$D_{j,n-j,0,\ldots,0}\ ,\ E_{(0,0,\ldots,0,m-1)+\varepsilon_j}$$

$$\therefore\ D_{j,n-j,0,\ldots,0} - E_{(\underbrace{0,\ldots,0}_{n-1},m-1)+\varepsilon_j} = (qt)^{n-j}(D_{\gamma_0} - E_{\gamma_0})$$

$$\therefore (D_{\gamma_0} - E_{\gamma_0}) + \sum_{j=1}^{n-1}(qt)^{n-j}(D_{\gamma_0} - E_{\gamma_0}) = 0$$

i.e. $[1 + qt + (qt)^2 + \cdots + (qt)^{n-1}](D_{\gamma_0} - E_{\gamma_0}) = 0 \Leftrightarrow D_{\gamma_0} - E_{\gamma_0} = 0 \Leftrightarrow D_{\gamma_0} = E_{\gamma_0}$. $\square$

Corollary 4.3.4: For any $a_1,\ldots,a_\ell \in \mathbb{Z}$, we have

$$E_{a_\ell,\ldots,a_2,a_1}\cdot 1 = E_{a_\ell,\ldots,a_2,0}\cdot 1 \qquad (\text{independent of } a_1)$$

Proof: Suppose $a_i > 0\ \forall\ 1 \le i \le \ell$. Then $(a_\ell,\ldots,a_1)$ corresponds to an admissible path γ from $(0,\ell)$ to $(a_1+a_2+\cdots+a_\ell, 0)$.

Consider $\vec{b}(\gamma)$. Recall that Lemma 3.4.2 in Path (§4.1) we know $\underset{D_{\vec{b}(\gamma)}\cdot 1}{\overset{}{D_\gamma \cdot 1}}$ is independent of trailing zeros which corresponds to the # E steps (except the last E step) on $y=0$

(i.e. $a_1 - 1$ = # trailing zeros in $\vec{b}(\gamma)$)

$\because$ $D_\gamma = E_\gamma$ (by Prop 4.3.3)

$\therefore$ $E_\gamma \cdot 1$ is independent on the number of E-steps on $x=0$ as long as there is one E-step on $y=0$ (i.e. as long as $a_1 \ge 1$) to keep γ admissible.

Known: The symmetry of $\bar{\Phi}\varepsilon^{t}$: $f(x^{m,n}) \mapsto f(x^{m+kn,n})$ sends $E_{a_\ell,\dots,a_1}$ to $E_{a_\ell+k, a_{\ell-1}+k, \dots, a_1+k}$ (b/c $p_1[x^{a,1}] \mapsto p_1[x^{a+k,1}]$ which corresponds to $z_\ell^{a_\ell} \cdots z_1^{a_1} \longmapsto z_\ell^{a_\ell+k} \cdots z_1^{a_1+k}$)

By Lemma 3.4.1 (Path) (c.f. §3.3), $\nabla^{k} f(x^{m,n}) \nabla^{-k} = f(x^{m+kn,n}) \Rightarrow \nabla^{k} \cdot E_{a_\ell,\dots,a_1} = E_{a_\ell+k,\dots,a_1+k} \cdot \nabla^{k}$

$$\Rightarrow (\nabla^{k} \cdot E_{a_\ell \cdots a_1}) \cdot 1 = (E_{a_\ell+k,\dots,a_1+k}\, \nabla^{k}) \cdot 1$$

$$\Rightarrow \nabla^{k} (E_{a_\ell \cdots a_1} \cdot 1) = E_{a_\ell+k,\dots,a_1+k} (\nabla^{k} \cdot 1)$$

$\nabla \cdot 1 = 1 \Rightarrow \nabla^{k} \cdot 1 = 1 \ \forall k$

$$\Rightarrow \nabla^{k} (E_{a_\ell \cdots a_1} \cdot 1) = E_{a_\ell+k,\dots,a_1+k} \cdot 1$$

$\therefore$ For any $a_1,\dots,a_\ell \in \mathbb{Z}^{\ell}$, we can find a large enough $k \geq 1$ such that $a_i + k \geq 0 \ \forall i$ with $a_1 + k \geq 1$.

Then by the result above, $E_{a_\ell+k,\dots,a_1+k} \cdot 1 = E_{a_\ell+k,\dots,a_2+k,\,k} \cdot 1$ ($k \geq 1$) $\Rightarrow E_{a_\ell,\dots,a_1} \cdot 1 = \nabla^{-k} (E_{a_\ell+k,\dots,a_2+k,k} \cdot 1)$

$$= E_{a_\ell,\dots,a_2,0} \cdot 1 \qquad \square$$

e.g. • $E_{0,1,1,0,0,1,0,1} \cdot 1 = \nabla^{-1} (E_{1,2,2,1,1,2,1,2} \cdot 1)$ all entries ≥ 0 with last entry ≥ 1 can be changed to 1 (or any (positive) integer)

$$= \nabla^{-1} (E_{1,2,2,1,1,2,1,1} \cdot 1)$$

$$= E_{0,1,1,0,0,1,0,0} \cdot 1$$

• $E_{-3,0,2,-4,1,3,-2} \cdot 1 = \nabla^{-5} (\underbrace{E_{2,5,7,1,6,8,3}}_{+5 \text{ from original}} \cdot 1) = \nabla^{-5} (E_{2,5,7,1,6,8,5} \cdot 1) = E_{-3,0,2,-4,1,3,0} \cdot 1$

we want to get 0 after "−5", hence choose 4