Def: Let $\alpha, \beta \in \mathbb{Z}^\ell$ be GL_ℓ weights and $\sigma \in S_\ell$. The LLT series $\mathcal{L}^{\sigma}_{\beta/\alpha}(x_1, \ldots, x_\ell; q)$ is the infinite formal sum of irreducible GL_ℓ characters in which the coeff of χ_λ is defined by

$$\langle \chi_\lambda \rangle \mathcal{L}^{\sigma^{-1}}_{\beta/\alpha}(x; q^{-1}) = \langle E^{\sigma}_{\beta} \rangle \chi_\lambda \cdot E^{\sigma}_{\alpha}$$

* Since coeff in $E^{\sigma}_{\lambda}(x;q)$ are in q^{-1}, coeff. in $\mathcal{L}^{\sigma}_{\beta/\alpha}(x;q)$ are thus polynomials in q.

The following theorem makes computation of $\mathcal{L}^{\sigma}_{\beta/\alpha}$ easier.

Prop: $\mathcal{L}^{\sigma}_{\beta/\alpha}(x;q) = H_q\left(w_0\left(F^{\sigma^{-1}}_{\beta}(x;q)\,\overline{E^{\sigma^{-1}}_{\alpha}(x;q)}\right)\right)$ where $w_0(i) = \ell+1-i$ $\forall i \in [\ell]$ (i.e. w_0: longest permutation in S_ℓ).

Proof: $\because$ $\{E^{\sigma^{-1}}_{\lambda}\}_{\lambda \in \mathbb{Z}^\ell}$ and $\{F^{\sigma^{-1}}_{\lambda}\}_{\lambda \in \mathbb{Z}^\ell}$ are dual bases

$\therefore$ coeff of $E^{\sigma^{-1}}_{\beta}(x;q^{-1})$ in $\chi_\lambda \cdot E^{\sigma^{-1}}_{\alpha}(x;q^{-1}) = \langle F^{\sigma^{-1}}_{\beta}(x^{-1};q), \chi_\lambda E^{\sigma^{-1}}_{\alpha}(x;q^{-1}) \rangle_{q^{-1}}$

$$= \langle x^0 \rangle \chi_\lambda F^{\sigma^{-1}}_{\beta}(x^{-1};q) E^{\sigma^{-1}}_{\alpha}(x;q^{-1}) \prod_{i<j} \frac{1-\frac{x_i}{x_j}}{1-q\frac{x_i}{x_j}}$$

$$w_0\left(\prod_{i<j} \frac{1-\frac{x_j}{x_i}}{1-q\frac{x_j}{x_i}}\right) = \prod_{i>j} \frac{1-\frac{x_j}{x_i}}{1-q\frac{x_j}{x_i}} = \prod_{i<j} \frac{1-\frac{x_i}{x_j}}{1-q\frac{x_i}{x_j}}$$

$x_i \mapsto x_i^{-1}$
Then take w_0
(constant term is not affected by these operations)

$$= \langle x^0 \rangle \overline{\chi_\lambda}\, w_0\left(F^{\sigma^{-1}}_{\beta}(x;q) E^{\sigma^{-1}}_{\alpha}(x^{-1};q^{-1})\right) \prod_{i<j} \frac{1-\frac{x_i}{x_j}}{1-q\frac{x_i}{x_j}}$$

$w_0(\overline{\chi_\lambda}) = \overline{\chi_\lambda}$ b/c χ_λ is symmetric

Recall §2 (c.f. Extended Delta notes §3.4 P.2 on Lemma 2.3.1)
we have
$\langle \chi_\lambda \rangle \sigma(\phi(x)) = \langle x^0 \rangle \overline{\chi_\lambda}\, \phi(x) \prod_{i<j}\left(1-\frac{x_i}{x_j}\right)$

$$= \langle \chi_\lambda \rangle \sigma\left(w_0\left(F^{\sigma^{-1}}_{\beta}(x;q) E^{\sigma^{-1}}_{\alpha}(x^{-1};q^{-1})\right) \prod_{i<j} \frac{1}{1-q\frac{x_i}{x_j}}\right) = \langle \chi_\lambda \rangle H_q\left(w_0\left(F^{\sigma^{-1}}_{\beta}(x;q)\,\overline{E^{\sigma^{-1}}_{\alpha}(x;q)}\right)\right). \quad \square$$

Take the expression and expand as a formal Laurent series in q, i.e. expand $\prod \frac{1}{1-q\frac{x_i}{x_j}} = \prod\left(1+q\frac{x_i}{x_j}+q^2\frac{x_i^2}{x_j^2}+\cdots\right)$
as a Laurent series and apply the Lemma in §2 term by term,
and recombine to get the expression $\sigma\left(w_0\left(F^{\sigma^{-1}}_{\beta}(x;q) E^{\sigma^{-1}}_{\alpha}(x^{-1};q^{-1})\right)\prod_{i<j}\frac{1}{1-q\frac{x_i}{x_j}}\right)$
because σ is linear

e.g. $w_0 = s_1 \in S_2$ as $\ell = 2$

$$\mathcal{L}^{s_1}_{32/11}(x;q) = H_q\left(w_0\left(F^{s_1}_{32}(x;q)\,\underbrace{\overline{E^{s_1}_{11}(x;q)}}_{E^{s_1}_{11} = x^{11} = x_1x_2}\right)\right)$$

$$= H_q\left(w_0\left(F^{s_1}_{32}(x_1,x_2;q)\, x_1^{-1}x_2^{-1}\right)\right)$$

$$\overline{E^{s_1 w_0}_{-3,-2}(x;q)} = \overline{E^{id}_{(-3,-2)}(x_1,x_2;q)} = \overline{q^{-1}T_1(x^{-2,-3})} = \overline{q^{-1}(q-1)x_1^{-2}x_2^{-3} + q^{-1}q\, x_1^{-3}x_2^{-2}} = (1-q)x_1^2x_2^3 + x_1^3x_2^2$$

$$= H_q\left(w_0\left((1-q)x_1x_2^2 + x_1^2x_2\right)\right)$$

$$= H_q\left((1-q)x_1^2x_2 + x_1x_2^2\right)$$

$$= \sum_{w \in S_2}\left(\frac{(1-q)x_1^2x_2 + x_1x_2^2}{\left(1-\frac{x_2}{x_1}\right)\left(1-q\frac{x_1}{x_2}\right)}\right)$$

$$= \frac{(1-q)x_1^2x_2 + x_1x_2^2}{\left(1-\frac{x_2}{x_1}\right)\left(1-q\frac{x_1}{x_2}\right)} + \frac{(1-q)x_1x_2^2 + x_1^2x_2}{\left(1-\frac{x_1}{x_2}\right)\left(1-q\frac{x_2}{x_1}\right)}$$

$$= \frac{1}{x_1-x_2}\left[\left((1-q)x_1^3x_2 + x_1^2x_2^2\right)\left(1+q\frac{x_1}{x_2}+q^2\frac{x_1^2}{x_2^2}+\cdots\right) - \left((1-q)x_1x_2^3 + x_1^2x_2^2\right)\left(1+q\frac{x_2}{x_1}+q^2\frac{x_2^2}{x_1^2}+\cdots\right)\right]$$

$$= \frac{1}{x_1-x_2}\left[(1-q)\left((x_1^3x_2 - x_1x_2^3) + q(x_1^4-x_2^4) + q^2\left(\frac{x_1^5}{x_2}-\frac{x_2^5}{x_1}\right) + q^3\left(\frac{x_1^6}{x_2^2}-\frac{x_2^6}{x_1^2}\right)+\cdots\right) + x_1^2x_2^2\left(q\frac{x_1^2-x_2^2}{x_1x_2} + q^2\frac{x_1^4-x_2^4}{x_1^2x_2^2}+\cdots\right)\right]$$

$$= (1-q)\left[\overset{\chi_{21}}{x_1x_2(x_1+x_2)} + q\,\overset{\chi_{30}}{(x_1^3+x_1^2x_2+x_1x_2^2+x_2^3)} + q^2\,\overset{\chi_{4-1}}{\left(\frac{x_1^5+x_1^4x_2+\cdots+x_2^5}{x_1x_2}\right)} + q^3\,\overset{\chi_{5-2}}{\left(\frac{x_1^7+x_1^6x_2+\cdots+x_2^8}{x_1^2x_2^2}\right)}+\cdots\right]$$

$$+\ q\left[x_1x_2(x_1+x_2) + q(x_1^3+x_1^2x_2+x_1x_2^2+x_2^3) + q^2\left(\frac{x_1^5+x_1^4x_2+\cdots+x_2^5}{x_1x_2}\right) + q^3\left(\frac{x_1^7+x_1^6x_2+\cdots+x_2^8}{x_1^2x_2^2}\right)+\cdots\right]$$

$$= \chi_{21} + q\chi_{30} + q^2\chi_{4-1} + q^3\chi_{5-2} + \cdots$$