Theorem 4.4.1 : For $0 \le \ell < m \le N$, we have

$$\left(\omega\left(h_\ell[B]\, e_{m-\ell-1}[B-1]\, e_{N-\ell}\right)\right)(x_1,\dots,x_m) = H^m_{q,t}\left(\frac{x_1x_2\cdots x_m}{\prod_{i=1}^{m-1}\left(1-qt\frac{x_i}{x_{i+1}}\right)}\, h_{N-m}(x_1,\dots,x_m)\, e_\ell(x_2^{-1},x_3^{-1},\dots,x_m^{-1})\right)_{pol}$$

Proof: For $f \in \Lambda$, set $g(X) := (\omega f)[X+\frac{1}{M}]$.

By Prop 3.3.1 (ii),

$$f(X^{1,0}) \text{ acts as } (\omega f)[B-\tfrac{1}{M}]$$

$$\underbrace{(\omega f)[X^{1,0}+\tfrac{1}{M}]}_{g(X^{1,0})} \text{ acts as } (\omega(\omega f))[B+\tfrac{1}{M}-\tfrac{1}{M}] = f[B]$$

Fact: Let $g[X+Y] = \sum g_{(1)}(X)\, g_{(2)}(Y)$. Then $g\xi = \sum \left((\mathrm{Ad}\, g_{(1)})\xi\right) g_{(2)}$

$$\therefore\ f[B]\xi\cdot 1 = g(X^{1,0})\xi\cdot 1 = \sum \left(\mathrm{Ad}\, g_{(1)}(X^{1,0})\,\xi\right) g_{(2)}(X^{1,0})\cdot 1$$

$\because\ g[X+Y] = (\omega f)[X+Y+\frac{1}{M}] = \sum \underbrace{(\omega f)_{(1)}[X]}_{g_{(1)}(X)}\ \underbrace{(\omega f)_{(2)}[Y+\frac{1}{M}]}_{g_{(2)}(Y)}$ and $h[B]\cdot 1 = h[0]\cdot 1$ for any $h \in \Lambda$

$$\therefore\ \sum \left(\mathrm{Ad}\, g_{(1)}(X^{1,0})\,\xi\right) g_{(2)}(X^{1,0})\cdot 1 = \sum \left(\mathrm{Ad}\left((\omega f)_{(1)}(X^{1,0})\right)\xi\right)(\omega f)_{(2)}[X^{1,0}+\tfrac{1}{M}]\cdot 1$$
$$= \sum \left(\mathrm{Ad}\left((\omega f)_{(1)}(X^{1,0})\right)\xi\right)(\omega f)_{(2)}[0]\cdot 1 = \left(\sum \left(\mathrm{Ad}\left((\omega f)_{(1)}(X^{1,0})\right)\xi\right)(\omega f)_{(2)}[0]\right)\cdot 1$$

Recall:

- $\left(\mathrm{Ad}(\omega f)(X^{1,0})\right)\xi = \sum \left(\mathrm{Ad}(\omega f)_{(1)}(X^{1,0})\,\xi\right)\left(\mathrm{Ad}(\omega f)_{(2)}(X^{1,0})\,1\right)$
- $(\mathrm{Ad}\, f)\xi = \sum_i f_i\, \xi\, g_i$ where $f[X-Y] = \sum f_i(X)\, g_i(Y)$ (c.f. §3.1–3.2) $\forall f \in \Lambda$

$\therefore\ (\mathrm{Ad}\, f(X))1 = \sum_i f_i(X)\, g_i(X) = f[X-X] = f[0]$ $\forall f \in \Lambda$

Sub $f(X)$ by $(\omega f)_{(2)}(X^{1,0})$

Hence $\left(\mathrm{Ad}(\omega f)(X^{1,0})\right)\xi = \sum \left(\mathrm{Ad}(\omega f)_{(1)}(X^{1,0})\,\xi\right)\omega f_{(2)}[0]$

$$\therefore\ \sum \left(\mathrm{Ad}\, g_{(1)}(X^{1,0})\,\xi\right) g_{(2)}(X^{1,0})\cdot 1 = \left(\sum \left(\mathrm{Ad}\left((\omega f)_{(1)}(X^{1,0})\right)\xi\right)(\omega f)_{(2)}[0]\right)\cdot 1 = \left(\left(\mathrm{Ad}(\omega f)(X^{1,0})\right)\xi\right)\cdot 1$$

i.e. $f[B]\xi\cdot 1 = \left(\mathrm{Ad}(\omega f)(X^{1,0})\right)\xi\cdot 1$

Let $n = N-\ell > 0$. Take $\xi = E_{a_n,\dots,a_1}$

$$\therefore\ f[B]\, E_{a_n a_{n-1}\cdots a_1}\cdot 1 = \left(\mathrm{Ad}(\omega f)(X^{1,0})\right) E_{a_n a_{n-1}\cdots a_1}\cdot 1$$

By Lemma 4.1.2

$$= \psi^{op}\left(\frac{(\omega(\omega f))(z_1,\dots,z_n)\cdot z_1^{a_1}\cdots z_n^{a_n}}{\prod_{i=1}^{n-1}\left(1-qt\frac{z_i}{z_{i+1}}\right)}\right)\cdot 1$$

$$= \psi^{op}\left(\frac{f(z_1,\dots,z_n)\, z_1^{a_1}\cdots z_n^{a_n}}{\prod_{i=1}^{n-1}\left(1-qt\frac{z_i}{z_{i+1}}\right)}\right)\cdot 1$$

If $f(z_1,\dots,z_n) = \sum c_{\vec r}\, z_1^{r_1} z_2^{r_2}\cdots z_n^{r_n}$ then $\psi^{op}\left(\frac{f(z_1,\dots,z_n)\, z_1^{a_1}\cdots z_n^{a_n}}{\prod\left(1-qt\frac{z_i}{z_{i+1}}\right)}\right) = \sum c_{\vec r}\, \psi^{op}\left(\frac{z_1^{a_1+r_1} z_2^{a_2+r_2}\cdots z_n^{a_n+r_n}}{\prod\left(1-qt\frac{z_i}{z_{i+1}}\right)}\right) = \sum c_{\vec r}\, E_{a_n+r_n,\dots,a_1+r_1}$

$E_{a_n+r_n,a_{n-1}+r_{n-1},\dots,a_1+r_1}$

$$= \left(f(z_n,z_{n-1},\dots,z_1)\,\Big|\, z_n^{r_n}\cdots z_1^{r_1} \mapsto E_{a_n+r_n,\dots,a_1+r_1}\right)\cdot 1$$

$$= f(z_n,z_{n-1},\dots,z_1)\Big|_{z_n^{r_n}\cdots z_1^{r_1} \mapsto E_{a_n+r_n,\dots,a_1+r_1}\cdot 1}$$

By Corollary 4.3.4, $E_{a_n+r_n,\dots,a_2+r_2,a_1+r_1}\cdot 1 = E_{a_n+r_n,\dots,a_2+r_2,a_1}\cdot 1$ (independent of the last entry)

Hence $f[B]\, E_{a_n\cdots a_1}\cdot 1 = \underbrace{f(z_n,\dots,z_2,1)}_{= f[z_n+z_{n-1}+\cdots+z_2+1]}\Big|_{z_n^{r_n}\cdots z_2^{r_2} \mapsto E_{a_n+r_n,\dots,a_2+r_2,a_1}\cdot 1}$ — (*)

$\therefore\ f[B-1]\, E_{a_n\cdots a_1}\cdot 1 = f[z_n+z_{n-1}+\cdots+z_2]\Big|_{z_n^{r_n}\cdots z_2^{r_2} \mapsto E_{a_n+r_n,\dots,a_2+r_2,a_1}\cdot 1}$ — (**)

Note that $E_{\underbrace{0,0,\dots,0}_{r}} = \Phi(D_{\underbrace{0,0,\dots,0}_{r}})$ and $D_{\underbrace{0,\dots,0}_{r}} = e_r[-MX^{1,0}]$ by Prop 3.6.1 (Path) (c.f. §4.1) (Put $m=1, n=0, k=r$. $b_i=0\ \forall 1\le i\le r$, path: $(0,0)$ to $(r,0)$)

$D_{b_1 b_2 \dots b_{km}} = e_k[-MX^{m,n}] \Rightarrow D_{\underbrace{0,0,\dots,0}_{r}} = e_r[-MX^{1,0}]$

$\therefore E_{\underbrace{0,\dots,0}_{n}} = \Phi(e_n[-MX^{1,0}]) = e_n[-MX^{0,1}] \Rightarrow E_{0,0,\dots,0}\cdot 1 = e_n[-MX^{0,1}]\cdot 1 \underset{\text{By Prop 3.3.1(iii)}}{=} e_n\left[-\frac{-MX}{M}\right]^{\bullet} 1 = e_n(X)$

Therefore, using (**) with $f = e_{k-1}$, we have

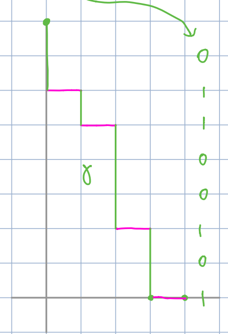
$$e_{k-1}[B-1]e_n(X) = e_{k-1}[B-1]\,E_{\underbrace{0,\dots,0}_{n}}\cdot 1 = e_{k-1}[z_n+\dots+z_2]\Big|_{z_n^{r_n}\cdots z_2^{r_2}\mapsto E_{r_n,r_{n-1},\dots,r_2,0}}\cdot 1 = \sum_{\substack{|I|=k-1\\ I\subseteq[n-1]}} E_{\varepsilon_I,0}\cdot 1 = \sum_I E_{\varepsilon_I,1}\cdot 1$$

Note that $(r_n,\dots,r_2)$ is a $(0,1)$-sequence with $k-1$ '1's

ε_I : $(0,1)$-sequence with 1 at position j iff $j\in I$, i.e. $\varepsilon_I = \sum_{j\in I}\varepsilon_j$

Note that $E_{\varepsilon_I,1}$ corresponds to an admissible path from $(0,n)$ to $(k,0)$ with a single E-step on $y=n-j$ for $j\in I$ and $x=0$ (b/c admissible)

e.g $E_{0,1,1,0,0,1,0,1}$: $n=8$, $k=4$, $I=\{2,3,6\}$, E-step on $x=6$, $x=5$, and $x=2$ (also $x=0$)



Hence $\displaystyle\sum_{\substack{|I|=k-1\\ I\subseteq[n-1]}} E_{\varepsilon_I,1}\cdot 1 = \sum_{\gamma\in P_{k,n}} E_{\vec a(\gamma)}\cdot 1$ where $P_{k,n} = \{\gamma : \gamma \text{ is admissible from } (0,n) \text{ to } (k,n) \text{ with } k-1 \text{ single E-steps}\}$

i.e $\vec a(\gamma)$ is a $(0,1)$-sequence with $k-1$ '1's

Apply (*) with $f = h_\ell$, we have

$$h_\ell[B]e_{k-1}[B-1]e_n(X) = h_\ell[B]\sum_{\gamma\in P_{k,n}} E_{\vec a(\gamma)}\cdot 1 = \sum_{\gamma\in P_{k,n}} h_\ell[B]E_{\vec a(\gamma)}\cdot 1 = \sum_{\gamma\in P_{k,n}}\left(\sum_{\substack{\beta\vDash\ell\\ \text{with } \ell(\beta)=n}} E_{\vec a(\gamma)+\beta}\cdot 1\right)$$

β tells us where to add E-steps on $\gamma\in P_{k,n}$ to form new admissible paths

Note that $E_{\vec a(\gamma)+\beta}$ corresponds to an admissible path $\tilde\gamma$ from $(0,n)$ to $(\ell+k,0)$ with $(\vec a(\gamma)+\beta)_j$ E-steps on $y=n-j$ $\forall 1\le j\le n$

For each fixed $\gamma\in P_{n,k}$, we have $k-1$ fixed single E-steps.

Let $j_1, j_2, \dots, j_{k-1} > 0$ s.t. γ has an E-step on $y=j_i$ $\forall 1\le i\le k-1$.

Then we can distinguish $\tilde\gamma$ by fixing all S-steps with the top lattice point at $y=j_i$ $\forall 1\le i\le k$.

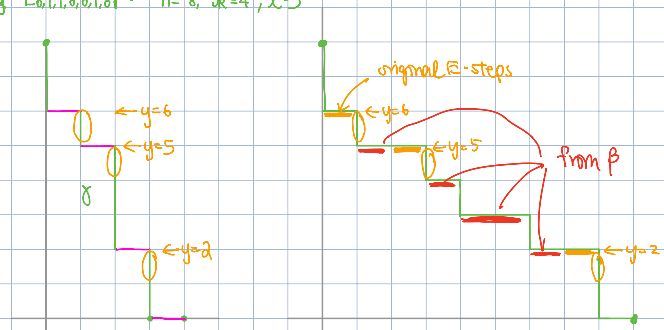
(because we can figure out what γ and β are)

The last entry of $\vec a(\gamma)$ must be positive

$-1)+1$

positive entries

e.g $E_{0,1,1,0,0,1,0,1}$: $n=8$, $k=4$, $\ell=5$



$\therefore \beta = (0,0,1,1,2,1,0,0)\vDash 5=\ell$

$\ell(\beta) = 8 = n$

so that the first entry of $(1,\dots,1)-\varepsilon_I$ is still 1.

$(\underbrace{1,1,\dots,1}_{k+\ell}) - \varepsilon_I$, $I\subseteq[2,k+\ell]$, $|I|=\ell$ (The position of 1's corresponds to the fixed S-steps obtained by γ.)

Hence $\vec b(\tilde\gamma)$ is obtained by : • an $(0,1)$-sequence of length $k+\ell$ with the first entry $=1$ and with $k-1$ more '1's

• $\vec s \in \mathbb{N}^{k+\ell}$ where $\vec s \vDash n-k$, corresponding to all other S-steps not fixed by γ

e.g. In the example above, $\vec{b}(\tilde{\gamma}) = (1,1,0,1,0,0,1,0,1) + (1,0,0,0,1,0,1,0,1)$ ← has an extra S-step on $y=0$

↑ has 1 S-step on $X=0$ other that the start one from $(0,8)$ to $(0,7)$

$$\therefore \sum_{\gamma \in P_{k,n}} \Big(\sum_{\substack{\beta \vDash \ell \\ \text{with } \ell(\beta)=n}} E_{\vec{a}(\gamma)+\beta} \cdot 1 \Big) = \sum_{\substack{\vec{s} \in \mathbb{N}^{k+\ell} \\ |\vec{s}| = n-k \\ I \subseteq [2,k+\ell],\ |I| = \ell}} D_{\vec{s} + \underbrace{(1,\dots,1)}_{k+\ell} - \varepsilon_I} \cdot 1$$

Set $m = k+\ell$

$$\text{i.e. } h_\ell[B]\, e_{m-\ell-1}[B-1]\, e_n(X) = \sum_{\substack{\vec{s} \in \mathbb{N}^m \\ |\vec{s}| = n-k \\ I \subseteq [2,m],\ |I| = \ell}} D_{\vec{s} + \underbrace{(1,1,\dots,1)}_{m} - \varepsilon_I} \cdot 1$$

$$\omega\big(h_\ell[B]\, e_{m-\ell-1}[B-1]\, e_n(X)\big) = \sum_{\substack{\vec{s} \in \mathbb{N}^m \\ |\vec{s}| = n-k \\ I \subseteq [2,m],\ |I| = \ell}} \omega\big(D_{\vec{s} + \underbrace{(1,\dots,1)}_{m} - \varepsilon_I} \cdot 1\big)(x_1, \dots, x_m)$$

By Prop 3.4.2

$$= \sum_{\substack{\vec{s} \in \mathbb{N}^m \\ |\vec{s}| = n-k \\ I \subseteq [2,m],\ |I| = \ell}} H^m_{q,t} \left(\frac{x^{\vec{s} + \overbrace{(1,\dots,1)}^{m} - \varepsilon_I}}{\prod_{i=1}^{m-1} \big(1 - qt \frac{x_i}{x_{i+1}}\big)} \right)_{pol}$$

$$= H^m_{q,t} \left(\frac{\sum_{\substack{\vec{s} \in \mathbb{N}^m \\ |\vec{s}| = n-k \\ I \subseteq [2,m],\ |I| = \ell}} x^{\vec{s} + \overbrace{(1,\dots,1)}^{m} - \varepsilon_I}}{\prod_{i=1}^{m-1} \big(1 - qt \frac{x_i}{x_{i+1}}\big)} \right)_{pol}$$

$$\because \sum_{\substack{\vec{s} \in \mathbb{N}^m \\ |\vec{s}| = n-k \\ I \subseteq [2,m],\ |I| = \ell}} x^{\vec{s} + \overbrace{(1,\dots,1)}^{m} - \varepsilon_I} = x_1 x_2 \cdots x_m \underbrace{h_{n-k}(x_1, \dots, x_m)}_{\text{corresponds to } x^{\vec{s}}} \underbrace{e_\ell(x_2^{-1}, x_3^{-1}, \dots, x_m^{-1})}_{\text{corresponds to } x^{-\varepsilon_I}}$$

$$\therefore \omega\big(h_\ell[B]\, e_{m-\ell-1}[B-1]\, e_n(X)\big) = H^m_{q,t} \left(\frac{x_1 \cdots x_m h_{n-k}(x_1, \dots, x_m)\, e_\ell(x_2^{-1}, \dots, x_m^{-1})}{\prod_{i=1}^{m-1} \big(1 - qt \frac{x_i}{x_{i+1}}\big)} \right)_{pol}$$

Sub $n = N - \ell$

$n-k = N-\ell-k = N-(\ell+k) = N-m$

$$\therefore \omega\big(h_\ell[B]\, e_{m-\ell-1}[B-1]\, e_{N-\ell}(X)\big) = H^m_{q,t} \left(\frac{x_1 \cdots x_m h_{N-m}(x_1, \dots, x_m)\, e_\ell(x_2^{-1}, \dots, x_m^{-1})}{\prod_{i=1}^{m-1} \big(1 - qt \frac{x_i}{x_{i+1}}\big)} \right)_{pol} \qquad \square$$

Remark 4.4.2: By Corollary 3.7.2 (Path) (c.f. §4.1), all terms s_λ in the Schur expansion of $\omega(D_{\vec{b}} \cdot 1)(X)$ have $\ell(\lambda) \le m$ $\forall \vec{b} \in \mathbb{Z}^m$.
Hence Theorem 4.4.1 determines $\omega(h_\ell[B]\, e_{m-\ell-1}[B-1]\, e_{N-\ell})$ by

$$\omega\big(h_\ell[B]\, e_{m-\ell-1}[B-1]\, e_n(X)\big) = \sum_{\substack{\vec{s} \in \mathbb{N}^m \\ |\vec{s}| = n-k \\ I \subseteq [2,m],\ |I| = \ell}} \omega\big(D_{\vec{s} + \underbrace{(1,\dots,1)}_{m} - \varepsilon_I} \cdot 1\big)(x_1, \dots, x_m)$$

even though the statement involves only m variables.