Def: Let $\vec{b} = (b_1, b_2, \ldots, b_\ell) \in \mathbb{Z}^\ell$, set

$$\phi(z) = \frac{z_1^{b_1} z_2^{b_2} \cdots z_\ell^{b_\ell}}{(1-qt\frac{z_1}{z_2})(1-qt\frac{z_2}{z_3})\cdots(1-qt\frac{z_{\ell-1}}{z_\ell})}.$$

Laurent poly. Not a Laurent poly.

By Negut (2014) (The shuffle algebra revisited), there exists a Laurent polynomial $\nu(z) = \nu(z_1, \ldots, z_\ell)$ s.t. $H^\ell_{q,t}(\nu(z)) = H^\ell_{q,t}(\phi(z))$ and $\nu(z)$ represents a well-defined element of the shuffle algebra S.

The Negut element $D_{\vec{b}}$ and the transposed Negut element $E_{\vec{a}}$ (where $\vec{a} := \mathrm{rev}(\vec{b}) = (b_\ell, b_{\ell-1}, \ldots, b_1)$) are defined by

$$D_{\vec{b}} = D_{b_1 b_2 \cdots b_\ell} := \psi(\nu(z)) \in \mathcal{E}^+$$

$$E_{\vec{a}} = E_{b_\ell b_{\ell-1} \cdots b_1} := \Phi(D_{\vec{b}}) = \psi^{op}(\nu(z)) \in \Phi\mathcal{E}^+$$

Recall:

Prop 3.4.1 (Schiffmann, Vasserot 2013) There is an algebra isomorphism $\psi : S \to \mathcal{E}^+$, $z^a \mapsto p_1[-MX^{1,a}]$ and anti-isomorphism $\psi^{op} := \Phi \circ \psi : S \to \Phi\mathcal{E}^+$, $z^a \mapsto p_1[-MX^{a,1}]$

* On monomials in m variables, representing elements of tensor degree m in S,

$$\psi(z_1^{a_1} z_2^{a_2} \cdots z_m^{a_m}) = p_1[-MX^{1,a_1}] \cdots p_1[-MX^{1,a_m}]$$

$$\psi^{op}(z_1^{a_1} z_2^{a_2} \cdots z_m^{a_m}) = p_1[-MX^{a_m,1}] \cdots p_1[-MX^{a_1,1}]$$

Prop 3.4.2 (Path Prop 3.5.2) Let $\phi(z) = \phi(z_1, z_2, \ldots, z_\ell)$ be a Laurent polynomial representing an element of tensor degree ℓ in S.

Let $\zeta = \psi(\phi(z)) \in \mathcal{E}^+$.

With $\mathcal{E}$ acting on Λ as in Prop 3.3.1, we have

$$\omega(\zeta \cdot 1)(z_1, z_2, \ldots, z_\ell) = H^\ell_{q,t}(\phi(z))_{pol}.$$

e.g. Set $\ell = 1$. Let $b \in \mathbb{Z}$.

Then $\phi(z) = z^b$. Hence we can take $\nu(z) = z^b$ and hence $D_{\vec{b}} = \psi(z_1^b) = p_1[-MX^{1,b}]$ and $E_{\vec{b}} = \psi^{op}(z_1^b) = p_1[-MX^{b,1}]$

* Hence $D_{\vec{b}}$ and $E_{\vec{b}}$ acts on Λ as D_b and E_b respectively when $\ell(\vec{b}) = 1$ (by Prop 3.3.3).

Negut identifies certain $D_{\vec{b}}$ as ribbon skew Schur functions. Here is a special case:

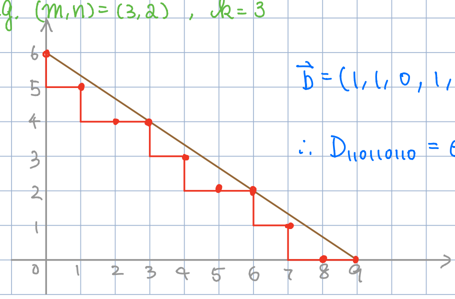
Prop. 3.6.1 (Path):
- $m, k \in \mathbb{Z}^+$, $n \in \mathbb{Z}_{\geq 0}$
- $\gcd(m,n) = 1$
- $b_i = \lceil \frac{n}{m} i \rceil - \lceil \frac{n}{m}(i-1) \rceil$ (# south steps at $x = i-1$ in the highest SE path weakly below $y = -\frac{n}{m}x + kn$)

i.e. x-intercept: $(km, 0)$, y-intercept: $(0, kn)$

Then

$$D_{b_1 b_2 \cdots b_{km}} = e_k[-MX^{m,n}]$$

e.g. $(m,n) = (3,2)$, $k = 3$



$\vec{b} = (1,1,0,1,1,0,1,1,0)$

$\therefore D_{110110110} = e_3[-MX^{3,2}]$

Lemma 3.6.2 (Path): For any indices $b_1,\dots,b_\ell \in \mathbb{Z}$, we have

$$D_{b_1,b_2,\dots,b_\ell,0}\cdot 1 = D_{b_1,b_2,\dots,b_\ell}\cdot 1$$ (i.e. we can omit trailing zeros when acting on 1)

* We have an E-version which we will prove later in §4.3

Proof: By Prop 3.4.2 (above), $\omega(D_{b_1,\dots,b_\ell}\cdot 1)(x_1,\dots,x_\ell) = H^{\ell}_{q,t}\left(\dfrac{x_1^{b_1}x_2^{b_2}\cdots x_\ell^{b_\ell}}{\prod_{i=1}^{\ell-1}(1-qt\frac{x_i}{x_{i+1}})}\right)_{pol}$ which is a linear combination of s_λ with $l(\lambda)\le \ell$.

$$\omega(D_{b_1,b_2,\dots,b_\ell,0}\cdot 1)(x_1,\dots,x_{\ell+1}) = H^{\ell+1}_{q,t}\left(\frac{x_1^{b_1}\cdots x_\ell^{b_\ell}x_{\ell+1}^{0}}{\prod_{i=1}^{\ell}(1-qt\frac{x_i}{x_{i+1}})}\right)_{pol}$$

$$= H^{\ell+1}_{q,t}\left(\frac{x_1^{b_1}\cdots x_\ell^{b_\ell}}{\prod_{i=1}^{\ell}(1-qt\frac{x_i}{x_{i+1}})}\right)_{pol}$$

$$= H^{\ell+1}_{q,t}\left(\frac{x_1^{b_1}\cdots x_\ell^{b_\ell}}{\prod_{i=1}^{\ell-1}(1-qt\frac{x_i}{x_{i+1}})}\cdot \Big(1+\underbrace{qt\frac{x_\ell}{x_{\ell+1}}+q^2t^2\frac{x_\ell^2}{x_{\ell+1}^2}+\cdots}\Big)\right)_{pol}$$

we can drop these because we only take "polynomial" parts ← i.e. x_λ with λ a poly. weight (partition)

which is a linear combination of Schur functions s_λ with $l(\lambda)\le\ell$.

$$\therefore\ \omega(D_{b_1,b_2,\dots,b_\ell,0}\cdot 1)(x_1,\dots,x_\ell) = \omega(D_{b_1,\dots,b_\ell}\cdot 1)(x_1,\dots,x_\ell)$$ □

Hence a corollary to Prop 3.6.1 is $D_{b_1,\dots,b_{km},b_{km+1}}\cdot 1 = e_k[-MX^{m,n}]\cdot 1$

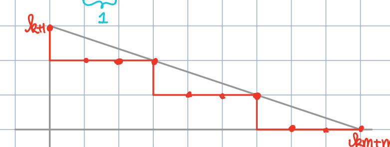
this is always 0.

Also, in the example above, $D_{1,1,0,1,1,0,1,1,0,0}\cdot 1 = e_3[-MX^{3,2}]\cdot 1$ and $D_{1,1,0,1,1,0,1,1}\cdot 1 = e_3[-MX^{3,2}]\cdot 1$

Corollary: If $n=1$, we have $\nabla^m e_k(x) = e_k[-MX^{m,1}]\cdot 1$.

Proof: $e_k[-MX^{0,1}]\cdot 1 \overset{\text{By Prop 3.3.1 (iii)}}{=} e_k(x)$ and $\nabla(1)=1$

$$\therefore\ e_k[-MX^{m,1}] = \nabla^m e_k[-MX^{0,1}]\nabla^{-m} \Rightarrow e_k[-MX^{m,1}]\cdot 1 = \nabla^m e_k[-MX^{0,1}]\underbrace{\nabla^{-m}\cdot 1}_{1} = \nabla^m e_k[-MX^{0,1}]\cdot 1 = \nabla^m e_k(x)$$ □



* Prop 3.4.2 implies $(\omega\nabla^m e_{k+1})(x_1,\dots,x_\ell) = \omega(e_{k+1}[-MX^{m,1}]\cdot 1)(x_1,\dots,x_\ell)$

$$= \omega(D_{\vec{b}}\cdot 1)(x_1,\dots,x_\ell)$$

$$= H^{\ell}_{q,t}\left(\frac{x^{\vec{b}}}{\prod_{i=1}^{\ell-1}(1-qt\frac{x_i}{x_{i+1}})}\right)_{pol}$$

$\vec{b} = (\underbrace{1,\overbrace{0,0,0\dots 0}^{m-1}},\underbrace{1,0,0,0,\dots},\underbrace{1,0,0,\dots 0},\dots,\underbrace{1,0,0,\dots 0})$

k "$1,0,0,\dots,0$"s

$b_1=b_{m+1}=b_{2m+1}=\cdots=b_{km+1}=1$ and all other b_j's are 0.

raising operator formula →

$$= \sigma\left(\frac{x_1x_{m+1}x_{2m+1}\cdots x_{km+1}\prod_{i+1<j}(1-qt\frac{x_i}{x_j})}{\prod_{1\le i<j\le\ell}(1-q\frac{x_i}{x_j})(1-t\frac{x_i}{x_j})}\right)_{pol}$$

$$= \left(\sum_{w\in S_\ell} w\left(\frac{x_1x_{m+1}x_{2m+1}\cdots x_{km+1}\prod_{i+1<j}(1-qt\frac{x_i}{x_j})}{\prod_{1\le i<j\le\ell}(1-q\frac{x_i}{x_j})(1-t\frac{x_i}{x_j})(1-\frac{x_j}{x_i})}\right)\right)_{pol}$$ for any $\ell\ge km+1$.

b/c we only take "pol"

Useful identities: (1) $D_{b_1,\dots,b_\ell}D_{b_{\ell+1},\dots,b_n} = D_{b_1,b_2,\dots,b_n} - qt\,D_{b_1,\dots,b_\ell+1,b_{\ell+1}-1,b_{\ell+2},\dots,b_n}$

for any indices $a_1,\dots,a_n,b_1,\dots,b_n\in\mathbb{Z}$.

(2) $E_{a_n,\dots,a_{\ell+1}}E_{a_\ell,\dots,a_1} = E_{a_n,\dots,a_1} - qt\,E_{a_n,\dots,a_{\ell+1}-1,a_\ell+1,a_{\ell-1},\dots,a_1}$

e.g. We can write $D_{3,-2}\cdot 1$ in terms of $(D_a D_b \cdot 1)$'s and $(D_a \cdot 1)$'s (i.e. in terms of D's with indices in $\mathbb{Z}$ instead of vectors in $\mathbb{Z}^d$ with $d \geq 2$)

$$D_{2,-1} - qt D_{3,-2} = D_2 D_{-1} \Rightarrow D_{3,-2} = \frac{1}{qt}(D_{2,-1} - D_2 D_{-1})$$

Similarly, $D_{2,-1} = \frac{1}{qt}(D_{1,0} - D_1 D_0)$

$$\therefore D_{3,-2} = \frac{1}{q^2t^2} D_{1,0} - \frac{1}{q^2t^2} D_1 D_0 - \frac{1}{qt} D_2 D_{-1}$$

Hence
$$D_{3,-2}\cdot 1 = \frac{1}{q^2t^2} D_{1,0}\cdot 1 - \frac{1}{q^2t^2} D_1 D_0 \cdot 1 - \frac{1}{qt} D_2 D_{-1} \cdot 1$$
$$= \frac{1}{q^2t^2} D_1 \cdot 1 - \frac{1}{q^2t^2} D_1 D_0 \cdot 1 - \frac{1}{qt} D_2 D_{-1} \cdot 1$$
$$\updownarrow$$
$$\frac{1}{q^2t^2} p_1[-MX^{1,1}]\cdot 1 - \frac{1}{q^2t^2} p_1[-MX^{1,1}]\, p_1[-MX^{1,0}]\cdot 1 - \frac{1}{qt} p_1[-MX^{1,2}]\, p_1[-MX^{1,-1}]\cdot 1$$

Lemma 4.1.2. Let $\phi(z)$ be a Laurent polynomial representing an element of tensor degree n in S. Then

$$(\mathrm{Ad}\, f(X^{1,0}))\, \psi^{op}(\phi(z)) = \psi^{op}((\omega f)(z_1, z_2, \dots, z_n)\cdot \phi(z))$$

In particular, we have

Take $\phi(z) = \dfrac{z_1^{a_1}\cdots z_n^{a_n}}{\prod_{i=1}^{n-1}(1 - qt\frac{z_i}{z_{i+1}})}$

$$(\mathrm{Ad}\, f(X^{1,0}))\, E_{a_n,\dots,a_1} = \psi^{op}\left(\frac{(\omega f)(z_1,\dots,z_n)\cdot z_1^{a_1} z_2^{a_2}\cdots z_n^{a_n}}{\prod_{i=1}^{n-1}(1 - qt\frac{z_i}{z_{i+1}})}\right).$$

e.g.
$$\mathrm{Ad}\, f(X^{1,0})\, \psi^{op}(z_1^a z_2^b) = \mathrm{Ad}\, f(X^{1,0})\big(p_1[-MX^{b,1}]\, p_1[-MX^{a,1}]\big)$$
$$= \sum \Big((\mathrm{Ad}\, f_{(1)}(X^{1,0}))\, \overbrace{p_1[-MX^{b,1}]}^{-Mp_1(X^{b,1})}\Big)\Big((\mathrm{Ad}\, f_{(2)}(X^{1,0}))\, \overbrace{p_1[-MX^{a,1}]}^{-Mp_1(X^{a,1})}\Big)$$
$$= \sum (-M)^2 (\omega f_{(1)})[z]\Big|_{z^k \mapsto p_1(X^{b+k,1})}\, \omega f_{(2)}[z]\Big|_{z^k \mapsto p_1(X^{a+k,1})}$$

Change variable   change variable

$$= \sum (\omega f_{(1)})[z_2]\Big|_{z_2^k \mapsto p_1(-MX^{b+k,1})}\, \omega f_{(2)}[z_1]\Big|_{z_1^k \mapsto p_1(-MX^{a+k,1})}$$

Note: $g[z]\Big|_{z^k \mapsto p_1(-MX^{a+k,1})} = -M\, g[z]\Big|_{z^k \mapsto p_1(X^{a+k,1})}$

e.g. $z^2 + z^3\Big|_{z^k \mapsto p_1[-MX^{a+k,1}]}$
$= p_1[-MX^{a+2,1}] + p_1[-MX^{a+3,1}]$
$= -Mp_1(X^{a+2,1}) - Mp_1(X^{a+3,1})$
$= -M(z^2 + z^3)\Big|_{z^k \mapsto p_1(X^{a+k,1})}$

same reasoning as the case with z^b

$$= \sum \big(z_2^b\, \omega f_{(1)}[z_2]\big)\Big|_{z_2^k \mapsto p_1(-MX^{k,1})}\, z_1^a\, \omega f_{(2)}[z_1]\Big)\Big|_{z_1^k \mapsto p_1(-MX^{k,1})}$$

Note: for any $g \in \Lambda$
$g[z]\Big|_{z^k \mapsto p_1(X^{b+k,1})}$
$= (z^b g[z])\Big|_{z^k \mapsto p_1(X^{k,1})}$

e.g. $g[z] = z^3$
Then $g[z]\Big|_{z^k \mapsto p_1(X^{b+k,1})} = p_1(X^{b+3,1})$
$(z^b g[z])\Big|_{z^k \mapsto p_1(X^{k,1})}$
$= (z^{b+3})\Big|_{z^k \mapsto p_1(X^{k,1})} = p_1(X^{b+3,1})$

$$= \Big(z_1^a z_2^b\, \omega \sum f_{(1)}[z_2]\, f_{(2)}[z_1]\Big)\Big|_{z_1^k, z_2^k \mapsto p_1(-MX^{k,1})}$$
$$= \big(z_1^a z_2^b\, (\omega f)(z_2, z_1)\big)\Big|_{z_1^k, z_2^k \mapsto p_1(-MX^{k,1})}$$
$$= \psi^{op}((\omega f)(z_1, z_2)\, z_1^a z_2^b).$$

(The proof basically follows the same idea.)