Notation :

$$\sum_{i=a}^{b\,\#} f_i = \begin{cases} \sum_{i=a}^{b} f_i & \text{if } a \le b+1 \\ -\sum_{i=b+1}^{a-1} f_i & \text{if } a \ge b+1 \end{cases} = \sum_{i=a}^{\infty} f_i - \sum_{i=b+1}^{\infty} f_i$$

(so the indices are : $[a,\infty) - [b+1,\infty)$)

e.g.

- $\sum_{i=3}^{5\,\#} f_i = \sum_{i=3}^{5} f_i = f_3+f_4+f_5$
- $\sum_{i=3}^{2\,\#} f_i = \sum_{i=3}^{2} f_i = 0$ and $\sum_{i=3}^{2} f_i = -\sum_{i=3}^{2} f_i = 0$ (using the lower formula)

∴ "-2", -"3", -"4" are left, giving $-f_2, -f_3, -f_4$

- $\sum_{i=5}^{1\,\#} f_i = -\sum_{i=2}^{4} f_i = -f_2-f_3-f_4$ (indices are : "$[5,\infty) - [2,\infty)$")

Prop 4.2.1 : For any $a \in \mathbb{Z}$ and $\vec{b} = (b_1,\dots,b_\ell) \in \mathbb{Z}^\ell$, we have

(i) $[D_a, D_{b_1,\dots,b_\ell}] = -M \sum_{i=1}^{\ell} \sum_{k=a+1}^{b_i\,\#} D_{b_1,\dots,b_{i-1},k,a+b_i-k,b_{i+1},\dots,b_\ell}$

(ii) $[E_{b_\ell,\dots,b_1}, E_a] = -M \sum_{i=1}^{\ell} \sum_{k=a+1}^{b_i\,\#} E_{b_\ell,\dots,b_{i+1},a+b_i-k,k,b_{i-1},\dots,b_1}$

$-\sum_{k=3}^{3}$ i.e. k=3 only   k=0,1,2,3 D → -D

e.g. $[D_3, D_{2,-1}] = -(1-q)(1-t)\left(-\sum_{k=4}^{2\,\#} D_{k,3+2-k,-1} + \sum_{k=4}^{-1\,\#} D_{2,k,3-1-k}\right)$

$= -(1-q)(1-t)\left(-D_{3,2,-1} - D_{2,0,2} - D_{2,1,1} - D_{2,2,0} - D_{2,3,-1}\right) = (1-q)(1-t)\left(D_{3,2,-1} + D_{2,0,2} + D_{2,1,1} + D_{2,0,0} + D_{2,3,-1}\right)$

k=4,5

$[D_3, D_{2,-1,5}] = -(1-q)(1-t)\left(\sum_{k=4}^{2\,\#} D_{k,3+2-k,-1,5} + \sum_{k=4}^{-1\,\#} D_{2,k,3-1-k,5} + \sum_{k=4}^{5\,\#} D_{2,-1,k,3+5-k}\right)$

$= -(1-q)(1-t)\left(D_{3,2,-1,5} - D_{2,0,2,5} - D_{2,1,1,5} - D_{2,2,0,5} - D_{2,3,-1,5} + D_{2,-1,4,4} + D_{2,-1,5,3}\right)$

write all indices in the reverse order

$[E_{-1,2}, E_3] = -(1-q)(1-t)\left(E_{-1,2,3} - E_{2,0,2} - E_{1,1,2} - E_{0,2,2} - E_{-1,3,2}\right)$

To prove the proposition, we need the following notation and Lemma :

(1) Recall $\Omega[a_1+a_2+\cdots-b_1-b_2-\cdots] = \prod_{i=1}^{\infty} \frac{1-b_i}{1-a_i}$

$\therefore \Omega[Mz] = \frac{(1-qz)(1-tz)}{(1-z)(1-qtz)} = 1 - \frac{M}{(1-\frac{1}{z})(1-qtz)}$ and $\Omega[-Mz] = \frac{(1-z)(1-qtz)}{(1-qz)(1-tz)}$

(2) For any $f(z) = f(z_1, z_2, \dots, z_m)$ antisymmetric in z_i and z_{i+1} for some $1 \le i \le m-1$, we have

$$H^m_{q,t}\left(\Omega\left[M\frac{z_i}{z_{i+1}}\right] f(z)\right) = 0$$

Proof: $H^m_{q,t}\left(\Omega\left[M\frac{z_i}{z_{i+1}}\right] f(z)\right) = \sum_{w\in S_m} w\left(f(z)\frac{(1-q\frac{z_i}{z_{i+1}})(1-t\frac{z_i}{z_{i+1}})}{(1-\frac{z_i}{z_{i+1}})(1-qt\frac{z_i}{z_{i+1}})} \cdot \prod_{1\le j<k\le m} \frac{1-qt\frac{z_j}{z_k}}{(1-\frac{z_k}{z_j})(1-q\frac{z_j}{z_k})(1-t\frac{z_j}{z_k})}\right)$

$$= \sum_{w\in S_m} w\left(f(z)\boxed{\frac{(1-q\frac{z_i}{z_{i+1}})(1-t\frac{z_i}{z_{i+1}})}{(1-\frac{z_i}{z_{i+1}})(1-qt\frac{z_i}{z_{i+1}})}} \cdot \underbrace{\prod_{1\le j<k\le m}\frac{1}{1-\frac{z_k}{z_j}} \cdot \prod_{1\le j<k\le m}\frac{1}{1-\frac{z_j}{z_k}}}_{\prod_{\substack{1\le r,s\le m\\ r\ne s}}\frac{1}{1-\frac{z_r}{z_s}}} \cdot \prod_{1\le j<k\le m} \overset{\Omega\left[-M\frac{z_j}{z_k}\right]}{\overset{\parallel}{\boxed{\frac{(1-qt\frac{z_j}{z_k})(1-\frac{z_j}{z_k})}{(1-q\frac{z_j}{z_k})(1-t\frac{z_j}{z_k})}}}}\right)$$

cancels with j=i, k=i+1

$$= \sum_{w\in S_m} w\left(f(z)\cdot \overbrace{\prod_{\substack{1\le r,s\le m\\ r\ne s}}\frac{1}{1-\frac{z_r}{z_s}}}^{\text{invariant under } w\in S_m} \cdot \overbrace{\prod_{\substack{1\le j<k\le m\\ (j,k)\ne(i,i+1)}} \Omega\left[-M\frac{z_j}{z_k}\right]}^{\text{invariant under } s_i}\right) = 0$$

There is a bijection between $\{\sigma\in S_m : \ell(\sigma s_i) = \ell(\sigma)+1\}$ and $\{\sigma\in S_m : \ell(\sigma s_i) = \ell(\sigma)-1\}$

via the map $\gamma \leftrightarrow \gamma s_i$

Hence all elements are of the form σ or σs_i with $\ell(\sigma s_i) = \ell(\sigma)+1$

i.e. We can split S_m as $\{\sigma\in S_m : \ell(\sigma s_i) = \ell(\sigma)+1\} \sqcup \{\sigma s_i : \sigma\in S_m, \ell(\sigma s_i) = \ell(\sigma)+1\}$

Hence LHS $= \sum_{\substack{w\in S_m\\ \ell(ws_i)=\ell(w)+1}} (w+ws_i)\left(f(z)\prod_{r\ne s}\frac{1}{1-\frac{z_r}{z_s}} \prod_{\substack{j<k\\ (j,k)\ne(i,i+1)}} \Omega\left[-M\frac{z_j}{z_k}\right]\right)$

$= \prod_{r\ne s}\frac{1}{1-\frac{z_r}{z_s}} \sum_{\substack{w\in S_m\\ \ell(ws_i)=\ell(w)+1}} \underbrace{\left(wf(z) + ws_i f(z)\right)}_{0} w\left(\prod_{j<k}\Omega\left[-M\frac{z_j}{z_k}\right]\right)$

$= 0 = $ R.H.S.

Proof of Prop 4.2.1:

We first prove $D_aD_{\vec{b}}-D_{\vec{b}}D_a+M\sum_{i=1}^{\ell}\sum_{k=a+1}^{b_i}{}^{\#}D_{b_1,\dots,b_{i-1},k,a+b_i-k,b_{i+1},\dots,b_\ell}=0$ —— (*)

Note that if we can show that a rational function representing the L.H.S is in ker $H^{\ell+1}_{qt}$, then with the isomorphism $\Upsilon: S\to\mathcal{E}^+$, (*) holds.

Hence we can choose the rational functions $\phi(z)$ that Negut found (defined in § 4.1).

Let
$$\phi(\hat{z}_i):=\phi(z_1,\dots,z_{i-1},z_{i+1},\dots,z_{\ell+1})=\frac{z_1^{b_1}\cdots z_{i-1}^{b_{i-1}}z_{i+1}^{b_i}\cdots z_{\ell+1}^{b_\ell}}{(1-qt\frac{z_1}{z_2})(1-qt\frac{z_2}{z_3})\cdots(1-qt\frac{z_{i-2}}{z_{i-1}})(1-qt\frac{z_{i-1}}{z_{i+1}})(1-qt\frac{z_{i+1}}{z_{i+2}})\cdots(1-qt\frac{z_\ell}{z_{\ell+1}})}=\frac{z_1^{b_1}\cdots z_{i-1}^{b_{i-1}}z_{i+1}^{b_i}\cdots z_{\ell+1}^{b_\ell}}{(1-qt\frac{z_{i-1}}{z_{i+1}})\prod_{\substack{1\le j\le\ell\\ j\ne i-1,i}}(1-qt\frac{z_j}{z_{j+1}})}$$

For D_a: the rational function is z^a

$$\therefore\ \underset{\text{use } z_1}{D_a}\,\underset{\text{start with } z_2}{D_{\vec{b}}}=z_1^a\,\phi(\hat{z}_1)\quad\text{and}\quad D_{\vec{b}}D_a=\phi(\hat{z}_{\ell+1})\,z_{\ell+1}^a$$

Hence we want show:

$$H^{\ell+1}_{qt}\left(z_1^a\phi(\hat{z}_1)-\phi(\hat{z}_{\ell+1})z_{\ell+1}^a+M\frac{\sum_{i=1}^{\ell}\sum_{k=a+1}^{b_i}{}^{\#}z_1^{b_1}\cdots z_{i-1}^{b_{i-1}}z_i^k z_{i+1}^{a+b_i-k}z_{i+2}^{b_{i+1}}\cdots z_{\ell+1}^{b_\ell}}{\prod_{j=1}^{\ell}(1-qt\frac{z_j}{z_{j+1}})}\right)=0$$

Note that $z_i^a\phi(\hat{z}_i)-\phi(\hat{z}_{i+1})z_{i+1}^a$ is antisymmetric in z_i and z_{i+1}, Lemma 4.2.2 implies

$$H^{\ell+1}_{qt}\left(\Omega[M\tfrac{z_i}{z_{i+1}}](z_i^a\phi(\hat{z}_i)-\phi(\hat{z}_{i+1})z_{i+1}^a)\right)=0\quad\text{and hence}\quad\sum_{i=1}^{\ell}H^{\ell+1}_{qt}\left(\Omega[M\tfrac{z_i}{z_{i+1}}](z_i^a\phi(\hat{z}_i)-\phi(\hat{z}_{i+1})z_{i+1}^a)\right)=0\Rightarrow H^{\ell+1}_{qt}\left(\sum_{i=1}^{\ell}\Omega[M\tfrac{z_i}{z_{i+1}}](z_i^a\phi(\hat{z}_i)-\phi(\hat{z}_{i+1})z_{i+1}^a)\right)=0$$

$$\therefore\ H^{\ell+1}_{qt}\left(\sum_{i=1}^{\ell}\left(1-\frac{M}{(1-\frac{z_{i+1}}{z_i})(1-qt\frac{z_i}{z_{i+1}})}\right)(z_i^a\phi(\hat{z}_i)-\phi(\hat{z}_{i+1})z_{i+1}^a)\right)=0$$

$$\text{i.e. } H^{\ell+1}_{qt}\left(z_1^a\phi(\hat{z}_1)-\phi(\hat{z}_{\ell+1})z_{\ell+1}^a-M\sum_{i=1}^{\ell}\frac{z_i^a\phi(\hat{z}_i)-\phi(\hat{z}_{i+1})z_{i+1}^a}{(1-\frac{z_{i+1}}{z_i})(1-qt\frac{z_i}{z_{i+1}})}\right)=0$$

It suffices to show
$$\sum_{i=1}^{\ell}\frac{z_i^a\phi(\hat{z}_i)-\phi(\hat{z}_{i+1})z_{i+1}^a}{(1-\frac{z_{i+1}}{z_i})(1-qt\frac{z_i}{z_{i+1}})}=-\frac{\sum_{i=1}^{\ell}\sum_{k=a+1}^{b_i}{}^{\#}z_1^{b_1}\cdots z_{i-1}^{b_{i-1}}z_i^k z_{i+1}^{a+b_i-k}z_{i+2}^{b_{i+1}}\cdots z_{\ell+1}^{b_\ell}}{\prod_{j=1}^{\ell}(1-qt\frac{z_j}{z_{j+1}})}\ ;$$

$$\sum_{i=1}^{\ell}\frac{z_i^a\phi(\hat{z}_i)-\phi(\hat{z}_{i+1})z_{i+1}^a}{(1-\frac{z_{i+1}}{z_i})(1-qt\frac{z_i}{z_{i+1}})}=\sum_{i=1}^{\ell}\frac{z_i^a\phi(\hat{z}_i)}{(1-\frac{z_{i+1}}{z_i})(1-qt\frac{z_i}{z_{i+1}})}-\sum_{i=1}^{\ell}\frac{z_{i+1}^a\phi(\hat{z}_{i+1})}{(1-\frac{z_{i+1}}{z_i})(1-qt\frac{z_i}{z_{i+1}})}$$

$$=\sum_{i=1}^{\ell}\frac{z_i^a\phi(\hat{z}_i)}{(1-\frac{z_{i+1}}{z_i})(1-qt\frac{z_i}{z_{i+1}})}-\sum_{i=2}^{\ell+1}\frac{z_i^a\phi(\hat{z}_i)}{(1-\frac{z_i}{z_{i-1}})(1-qt\frac{z_{i-1}}{z_i})}$$

$$=\frac{z_1^a\phi(\hat{z}_1)}{(1-\frac{z_2}{z_1})(1-qt\frac{z_1}{z_2})}+\sum_{i=2}^{\ell}\left[\frac{1}{(1-\frac{z_{i+1}}{z_i})(1-qt\frac{z_i}{z_{i+1}})}-\frac{1}{(1-\frac{z_i}{z_{i-1}})(1-qt\frac{z_{i-1}}{z_i})}\right]z_i^a\phi(\hat{z}_i)-\frac{z_{\ell+1}^a\phi(\hat{z}_{\ell+1})}{(1-\frac{z_{\ell+1}}{z_\ell})(1-qt\frac{z_\ell}{z_{\ell+1}})}$$

Set $z_0\to0$, $z_{\ell+2}\to\infty$
$$=\sum_{i=1}^{\ell+1}\left[\underset{\uparrow\ \text{This is 0 when } i=\ell+1}{\frac{1}{(1-\frac{z_{i+1}}{z_i})(1-qt\frac{z_i}{z_{i+1}})}}-\underset{\uparrow\ \text{This is 0 when } i=1}{\frac{1}{(1-\frac{z_i}{z_{i-1}})(1-qt\frac{z_{i-1}}{z_i})}}\right]z_i^a\phi(\hat{z}_i)$$

Verify:
$$\frac{1}{(1-\frac{z_{i+1}}{z_i})(1-qt\frac{z_i}{z_{i+1}})}-\frac{1}{(1-\frac{z_i}{z_{i-1}})(1-qt\frac{z_{i-1}}{z_i})}$$
$$=\frac{1}{(1-qt\frac{z_i}{z_{i+1}})(1-qt\frac{z_{i-1}}{z_i})}\left(\frac{1-qt\frac{z_{i-1}}{z_i}}{1-\frac{z_{i+1}}{z_i}}-\frac{1-qt\frac{z_i}{z_{i+1}}}{1-\frac{z_i}{z_{i-1}}}\right)$$
$$=\frac{1}{(1-qt\frac{z_i}{z_{i+1}})(1-qt\frac{z_{i-1}}{z_i})}\left[\left(\frac{1}{1-\frac{z_{i+1}}{z_i}}-\frac{1}{1-\frac{z_i}{z_{i-1}}}\right)-qt\boxed{\left(\frac{\frac{z_{i-1}}{z_i}}{1-\frac{z_{i+1}}{z_i}}-\frac{\frac{z_i}{z_{i+1}}}{1-\frac{z_i}{z_{i-1}}}\right)}\right]$$
$$=\frac{1-qt\frac{z_{i-1}}{z_{i+1}}}{(1-qt\frac{z_i}{z_{i+1}})(1-qt\frac{z_{i-1}}{z_i})}\left(\frac{1}{1-\frac{z_{i+1}}{z_i}}-\frac{1}{1-\frac{z_i}{z_{i-1}}}\right)$$

$$\boxed{\left(\frac{\frac{z_{i-1}}{z_i}}{1-\frac{z_{i+1}}{z_i}}-\frac{\frac{z_i}{z_{i+1}}}{1-\frac{z_i}{z_{i-1}}}\right)}=\frac{z_{i-1}}{z_{i+1}}\left(\frac{\frac{z_{i+1}}{z_i}}{1-\frac{z_{i+1}}{z_i}}-\frac{\frac{z_i}{z_{i-1}}}{1-\frac{z_i}{z_{i-1}}}\right)$$
$$\frac{z_{i+1}}{z_i}\Big/\left(1-\frac{z_{i+1}}{z_i}\right)=\frac{1}{1-\frac{z_{i+1}}{z_i}}-1\qquad\frac{z_i}{z_{i-1}}\Big/\left(1-\frac{z_i}{z_{i-1}}\right)=\frac{1}{1-\frac{z_i}{z_{i-1}}}-1$$
$$=\frac{z_{i-1}}{z_{i+1}}\left(\frac{1}{1-\frac{z_{i+1}}{z_i}}-\frac{1}{1-\frac{z_i}{z_{i-1}}}\right)$$

$$=\sum_{i=1}^{\ell+1}\frac{(1-qt\frac{z_{i-1}}{z_{i+1}})}{(1-qt\frac{z_{i-1}}{z_i})(1-qt\frac{z_i}{z_{i+1}})}\left(\frac{1}{1-\frac{z_{i+1}}{z_i}}-\frac{1}{1-\frac{z_i}{z_{i-1}}}\right)z_i^a\phi(\hat{z}_i)$$

$$\frac{z_1^{b_1}z_2^{b_2}\cdots z_{i-1}^{b_{i-1}}z_{i+1}^{b_i}\cdots z_{\ell+1}^{b_\ell}}{(1-qt\frac{z_1}{z_2})\cdots(1-qt\frac{z_{i-2}}{z_{i-1}})(1-qt\frac{z_{i-1}}{z_{i+1}})(1-qt\frac{z_{i+1}}{z_{i+2}})\cdots(1-qt\frac{z_\ell}{z_{\ell+1}})}$$

$$\frac{z_i^a\,\boxed{\phi(\hat{z}_i)}\,(1-qt\frac{z_{i-1}}{z_{i+1}})}{(1-qt\frac{z_{i-1}}{z_i})(1-qt\frac{z_i}{z_{i+1}})}=\frac{z_1^{b_1}z_2^{b_2}\cdots z_{i-1}^{b_{i-1}}z_i^a z_{i+1}^{b_i}z_{i+2}^{b_{i+1}}\cdots z_{\ell+1}^{b_\ell}}{\prod_{j=1}^{\ell}(1-qt\frac{z_j}{z_{j+1}})}$$

$$=\sum_{i=1}^{\ell+1}\frac{\boxed{\frac{z_i^a\phi(\hat{z}_i)(1-qt\frac{z_{i-1}}{z_{i+1}})}{(1-qt\frac{z_{i-1}}{z_i})(1-qt\frac{z_i}{z_{i+1}})}}-\frac{z_{i+1}^a\phi(\hat{z}_{i+1})(1-qt\frac{z_i}{z_{i+2}})}{(1-qt\frac{z_i}{z_{i+1}})(1-qt\frac{z_{i+1}}{z_{i+2}})}}{1-\frac{z_{i+1}}{z_i}}$$

$$=\frac{\sum_{i=1}^{\ell}z_1^{b_1}z_2^{b_2}\cdots z_{i-1}^{b_{i-1}}\left(\boxed{\frac{z_i^a z_{i+1}^{b_i}-z_i^{b_i}z_{i+1}^a}{1-\frac{z_{i+1}}{z_i}}}\right)z_{i+2}^{b_{i+1}}\cdots z_{\ell+1}^{b_\ell}}{\prod_{j=1}^{\ell}(1-qt\frac{z_j}{z_{j+1}})}$$

$$\frac{z_i^a z_{i+1}^{b_i}-z_i^{b_i}z_{i+1}^a}{1-\frac{z_{i+1}}{z_i}}=\frac{z_i^{a+1}z_{i+1}^{b_i}-z_i^{b_i+1}z_{i+1}^a}{z_i-z_{i+1}}$$
$$=\begin{cases}z_i^a z_{i+1}^{b_i}+z_i^{a-1}z_{i+1}^{b_i+1}+\cdots+z_i^{b_i+1}z_{i+1}^{a-1} & \text{if } a>b_i+1\\ 0 & \text{if } a=b_i+1\\ -(z_i^{b_i}z_{i+1}^a+z_i^{b_i-1}z_{i+1}^{a+1}+\cdots+z_i^{a+1}z_{i+1}^{b_i-1}) & \text{if } a<b_i+1\end{cases}$$
$$=\begin{cases}\sum_{k=b_i+1}^{a}z_i^k z_{i+1}^{a+b_i-k} & \text{if } a\geqslant b_i+1\\ -\sum_{k=a+1}^{b_i}z_i^k z_{i+1}^{a+b_i-k} & \text{if } a\leqslant b_i+1\end{cases}=-\sum_{k=a}^{b_i}{}^{\#}z_i^k z_{i+1}^{a+b_i-k}$$

$$=\frac{-\sum_{i=1}^{\ell}z_1^{b_1}z_2^{b_2}\cdots z_{i-1}^{b_{i-1}}\left(\sum_{k=a+1}^{b_i}{}^{\#}z_i^k z_{i+1}^{a+b_i-k}\right)z_{i+2}^{b_{i+1}}\cdots z_{\ell+1}^{b_\ell}}{\prod_{j=1}^{\ell}(1-qt\frac{z_j}{z_{j+1}})}$$

□