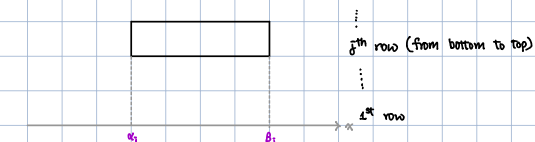
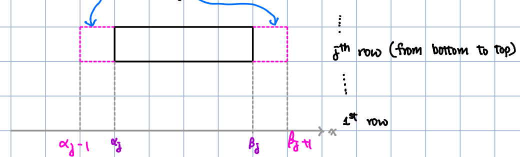
Def: For $\alpha, \beta \in \mathbb{Z}^{\ell}$ s.t. $\alpha_j \le \beta_j \ \forall\, 1 \le j \le \ell$, define

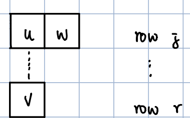
β/α = tuple of single row skew shapes β_j/α_j s.t. the left end has x-coordinate α_j and the right end has x-coordinate β_j.



"virtual" boxes adjacent to the left and right end of the row:

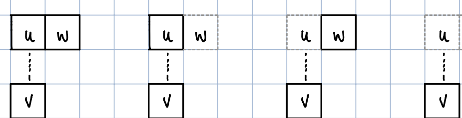


Def: Given a tuple of skew row shapes β/α, three boxes (u, v, w) form a w_0-triple when



(u, w can be "virtual" boxes)

Hence possible w_0-triples are



← This row is empty in the case. i.e. $\beta_j = \alpha_j$

Notation: $h_{w_0}(\beta/\alpha)$ = # w_0-triples in β/α (To count, we fix a "real" box and count the # left virtual or real boxes above it.)

i.e. Fix v

These are choices of u

These boxes must contain a box (may be virtual) on its right i.e. w exists

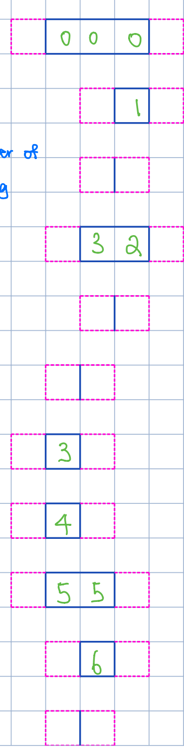
choices of v

By definition, $h_{w_0}(\beta/\alpha) = \sum_{1 \le r < j \le \ell} \left| [\alpha_r + 1, \beta_r] \cap [\alpha_j, \beta_j] \right|$

choices of u

e.g. $\beta = (1, 2, 2, 1, 1, 1, 2, 3, 2, 3, 3)$

$\alpha = (1, 1, 0, 0, 0, 1, 2, 1, 2, 2, 0)$



(number in each real box counts the number of left virtual / real box above it, i.e. counting choices of u to form a triple with v)

$\therefore\ h_{w_0}(\beta/\alpha)$ = sum of the numbers
$= 29$

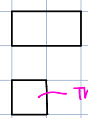
Def: For a totally ordered alphabet $\mathcal{A}$, a row strict tableau of shape β/α is a map

$S:\beta/\alpha \to \mathcal{A}$ s.t. each row is strictly increasing. Convention: $\begin{cases} S(x) = -\infty & \text{if } x \text{ is a left virtual box} \\ S(x) = +\infty & \text{if } x \text{ is a right virtual box} \end{cases}$

Notation: $RST(\beta/\alpha, \mathcal{A})$ = set of all row strict tableau of shape β/α in alphabet $\mathcal{A}$

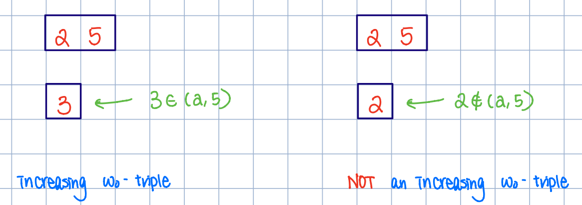
Set $RST(\beta/\alpha,\mathcal{A}) = \phi$ if $\alpha_j > \beta_j$ for some $1 \le j \le \ell$.

Def: A w_0-tuple (u,v,w) is an increasing w_0-triple in S if $S(u) < S(v) < S(w)$. $h_{w_0}(S)$ = # increasing w_0-triples in S.



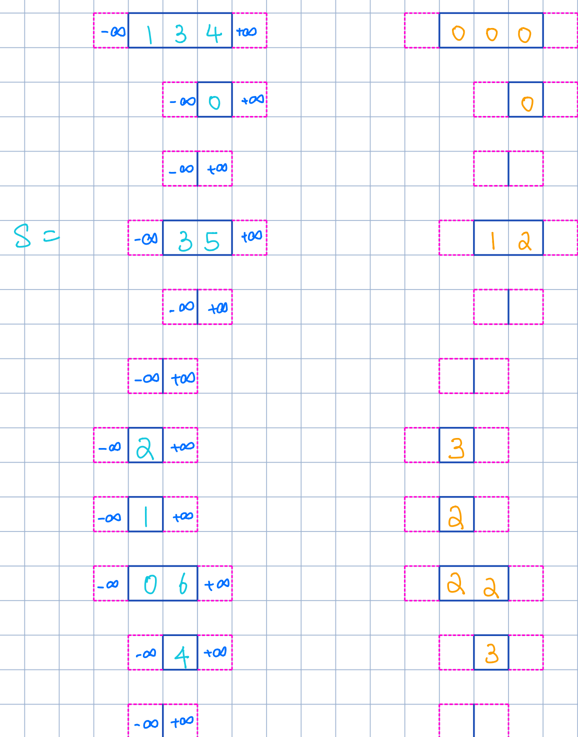
This value is in the "interval" formed by $(S(u), S(v))$

e.g.



e.g. $\beta = (1,2,2,1,1,1,2,3,2,3,3)$

$\alpha = (1,1,0,0,0,1,2,1,2,2,0)$



← The number in each real box (v) counts the number of 'u' forming an increasing w_0-triple with the fixed v box.

$\therefore h_{w_0}(S)$ = sum of these numbers $= 15$.

Def: For $S \in RST(\beta/\alpha, \mathbb{N})$, define

$$x^{wt_+(S)} := \prod_{\substack{b \in \beta/\alpha \\ S(b) \neq 0}} x_{S(b)}, \qquad x^{wt(S)} := \prod_{b \in \beta/\alpha} x_{S(b)}.$$

$x^{wt(S)} = x_0^{\#'0' \text{ in } S} \cdot x^{wt_+(S)}$

e.g. In the example above,

$$x^{wt_+(S)} = x_1^2 x_2 x_3^2 x_4^2 x_5 x_6, \qquad x^{wt(S)} = x_0^2 x_1^2 x_2 x_3^2 x_4^2 x_5 x_6 = x_0^2 x^{wt_+(S)}$$

Def : For $\alpha, \beta \in \mathbb{N}^m$, define

$$N_{\beta/\alpha} = N_{\beta/\alpha}(X;q) := \sum_{S \in RST(\beta/\alpha, \mathbb{Z}^+)} q^{h_{w_0}(S)} x^{wt(S)}$$

In this case $wt(S) = wt_+(S)$

* In Prop 4.5.2 (Path), $N_{\beta/\alpha} \in \Lambda$ with Schur expansion involving only s_λ with $\ell(\lambda) \le m$.