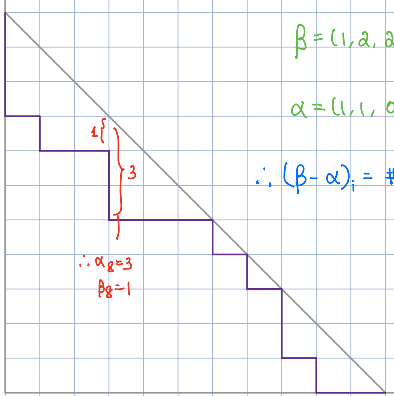
Def: The LLT data associated to a path $\lambda \in D_N$ is

$$\beta = (1, c_2(\lambda)+1, c_3(\lambda)+1, \dots, c_N(\lambda)+1)$$

$$\alpha = (c_2(\lambda), c_3(\lambda), c_4(\lambda), \dots, c_N(\lambda), 0)$$

where $c_i(\lambda)$ = area between λ and the line $x+y=N$ in col i from the right (between $x=N+1-i$ to $x=N-i$)

e.g. $N=11$



$\beta = (1,2,2,1,1,1,2,\boxed{3},2,3,3)$ ← bottom

vert. distance between λ and $x=N-i$

$\alpha = (1,1,0,0,0,1,2,\boxed{1},2,2,0)$ ← top

$\therefore (\beta-\alpha)_i$ = # South steps of λ on $x=N-i$.

2	2	1	2	1	0	0	0	1	1	0
c_1	c_{10}	c_9	c_8	c_7	c_6	c_5	c_4	c_3	c_2	c_1

Lemma 5.3.3: For $\alpha, \beta \in \mathbb{N}^N$ and $S \in RST(\beta/\alpha, \mathbb{N})$, let

- $I \subseteq [N]$: rows of S containing 0
- $T \in RST(\beta/\alpha+\varepsilon_I, \mathbb{Z}^+)$: tableau obtained by deleting all '0' boxes in S (must occur in the leftmost box in each row b/c rows are increasing) and entries in S are non-negative

Then

$$h_{w_0}(T) = h_{w_0}(S) - h_I(\alpha)$$

Proof: Let (u,v,w) be an increasing w_0-triple in S with u,w in row j and v in row r. Then $j>r$ and $S(u)<S(v)<S(w)$.

Case 1: $r \notin I$.

- If $j \notin I$, then (u,v,w) is also an increasing w_0-triple in S because $0<S(u)<S(v)<S(w)$
- If $j \in I$, then $S(u)=0$. Then u is a virtual box (adjacent to the left of row j) in T and hence $T(u)=-\infty$.
  i.e. (u,v,w) is still an increasing w_0-triple in S.

in S: α_j α_r Hence $\alpha_r = \alpha_j - 1$

Case 2: $r \in I$, i.e. $S(v)=0$ and hence $S(u)=-\infty$ (i.e. u is a virtual box in S) which also means v and w are the leftmost 'real' box in row r and row j of S resp.
Then $j \notin I$ because $S(w)>S(v)=0$. Removing box v from S to form T means (u,v,w) is not a w_0-increasing triple in T anymore.

$\therefore h_{w_0}(S) = \#\{(r,j): r<j,\ r\in I,\ j\notin I,\ \alpha_j=\alpha_r+1\}| + h_{w_0}(T) = h_I(\alpha) + h_{w_0}(T)$. Thus $h_{w_0}(T) = h_{w_0}(S) - h_I(\alpha)$. □

Lemma 5.3.2: For $\lambda \in D_N$ and its associated LLT data α, β, we have

$$\sum_{P \in L_{N,\ell}(\lambda)} q^{\operatorname{dinv}(P)} x^{wt_+(P)} = \sum_{\substack{I \subseteq [N-1] \\ |I|=\ell}} q^{h_I(\alpha)} N_{\beta/(\alpha+\varepsilon_I)}(X;q)$$

exactly ℓ zeros but no zero in the bottom row

Proof: There is a natural weight-preserving bijection

$P \in L_N(\lambda) \longleftrightarrow S \in RST(\beta/\alpha, \mathbb{N})$ (b/c $\beta-\alpha$ represents the lengths of South runs and labelling in P is strictly increasing
(labels of $x=i$ of P) $\longleftrightarrow$ (labels in row $N-i$) from bottom to top down each South runs)
(read top to bottom) (read left to right)

Claim: $\operatorname{dinv}(P) = h_{w_0}(S)$

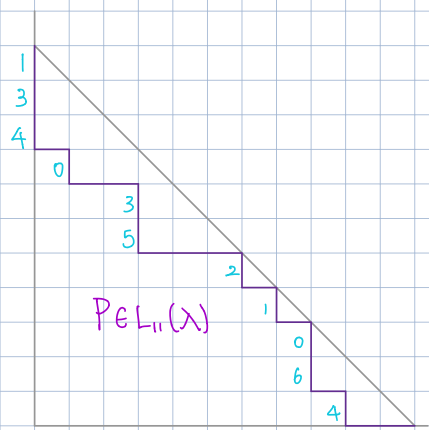
Order the alphabets $0>1>2>\cdots$ and call this $\hat{P}$ (same labelling as P with a different ordering in alphabets)

By Remark 2.2.3, $\operatorname{dinv}(P) = \operatorname{dinv}_1(\hat{P})$ The dinv defined in [13]: (HHLRU'05: A combinatorial formula for the character of the diagonal coinvariants)

By Prop 6.1.1 in Path, $\operatorname{dinv}_1(\hat{P}) = h_{w_0}(S)$ which completes the proof of the claim.

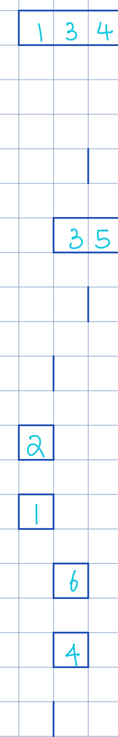
$$\therefore \sum_{P \in L_{N,\ell}(\lambda)} q^{\operatorname{dinv}(P)} x^{wt_+(P)} = \sum_{\substack{S \in RST(\beta/\alpha,\mathbb{N}) \\ S \text{ has exactly } \ell \text{ zeros, none in top row}}} q^{h_{w_0}(S)} x^{wt_+(S)} = \sum_{\substack{I \subseteq [N-1] \\ |I|=\ell}} \sum_{\substack{S \in RST(\beta/\alpha,\mathbb{N}) \\ 0 \text{ occurs in row } i \in I}} q^{h_{w_0}(S)} x^{wt_+(S)} = \sum_{\substack{I \subseteq [N-1] \\ |I|=\ell}} q^{h_I(\alpha)} \underbrace{\left(\sum_{T \in RST(\beta/(\alpha+\varepsilon_I),\mathbb{Z}^+)} q^{h_{w_0}(T)} x^{wt(T)} \right)}_{N_{\beta/(\alpha+\varepsilon_I)}(X;q)} = \sum_{\substack{I \subseteq [N-1] \\ |I|=\ell}} q^{h_I(\alpha)} N_{\beta/(\alpha+\varepsilon_I)}(X;q). \ \square$$

e.g.



$S =$

$T =$



$S \in RST(\beta/(\alpha, \mathbb{N}))$ $\quad$ $T \in RST(\beta/((\alpha+\varepsilon_{\{3,10\}}), \vec{I}))$

$\beta = (1, 2, 2, 1, 1, 1, 2, 3, 2, 3, 3)$

$\alpha = (1, 1, 0, 0, 0, 1, 2, 1, 2, 2, 0) \longrightarrow \alpha + \varepsilon_I = (1, 1, 1, 0, 0, 1, 2, 1, 2, 3, 0)$

$h_{w_0}(S) = 15$ $\quad$ $h_{w_0}(T) = 13$

$I = \{3, 10\}$

$h_I(\alpha) = 2$

$\alpha = (1, 1, 0, 0, 0, 1, 2, 1, 2, 2, 0)$

Def: Given $\vec{a} = (a_1, \ldots, a_{m-1}) \in \mathbb{N}^{m-1}$, $\tau = (\tau_1, \ldots, \tau_m) \in \mathbb{N}^m$, define

$\beta_{\vec{a}\tau} := (1, 2, \ldots, \tau_1+1, a_1+1, a_1+2, \ldots, a_1+\tau_2+1, a_2+1, a_2+2, \ldots, a_2+\tau_3+1, \ldots, a_{m-1}+1, a_{m-1}+2, \ldots, a_{m-1}+\tau_m+1)$

$\alpha_{\vec{a}\tau} := (1, 2, \ldots, \tau_1, a_1, a_1+1, a_1+2, \ldots, a_1+\tau_2, a_2, a_2+1, a_2+2, \ldots, a_2+\tau_3, a_3, \ldots, a_{m-1}+1, a_{m-1}+2, \ldots, a_{m-1}+\tau_m, 0)$ $\rightarrow$ (same as $\beta_{\vec{a}\tau}$ except: $a_1+\tau_2+1 \to a_2$, $a_2+\tau_3+1 \to a_3$, $\vdots$, $a_{m-2}+\tau_{m-1}+1 \to a_{m-1}$, $a_{m-1}+\tau_m+1 \to 0$)

* $\alpha_{\vec{a}\tau} + (1, \ldots, 1)$ and then shift entries once to the right (with last entry becomes the first entry) gives $\beta_{\vec{a}\tau}$.

e.g. $m = 7$, $\vec{a} = (1, 3, 0, 0, 1, 2)$, $\tau = (2, 3, 1, 1, 0, 2, 2)$

$(0, \vec{a}) + (1, 1, 1, 1, 1, 1, 1) = (1, 2, 4, 1, 1, 2, 3)$

$(0, \vec{a}) + \tau + (1, 1, 1, 1, 1, 1, 1) = (3, 5, 5, 2, 1, 4, 5)$

$\beta_{\vec{a}\tau} = (\underbrace{1, 2, 3}_{[1,3]}, \underbrace{2, 3, 4, 5}_{[2,5]}, 4, 5, 1, 2, 1, 2, 3, 4, 3, 4, 5)$

$\alpha_{\vec{a}\tau} = (1, 2, 1, 2, 3, 4, 3, 4, 0, 1, 0, 1, 2, 3, 2, 3, 4, 0)$

Lemma 5.3.6: For $0 \le \ell < m \le N$,

$$\langle z^{N-m}\rangle \sum_{\substack{\lambda\in D_N\\ P\in L_{N,\ell}(\lambda)}} t^{|\delta/\lambda|} \prod_{\substack{1<i\le N\\ c_i(\lambda)=c_{i-1}(\lambda)+1}} (1+zt^{-c_i(\lambda)})\, q^{\mathrm{dinv}(P)} x^{\mathrm{wt}_+(P)} = \sum_{\substack{I\subseteq[N-1]\\ |I|=\ell}} \sum_{\substack{\tau,(0,\vec a)\in\mathbb{N}^m\\ |\tau|=N-m}} t^{|\vec a|} q^{h_I(\alpha_{\vec a\tau})} N_{\beta_{\vec a\tau}/(\alpha_{\vec a\tau}+\varepsilon_I)}(X;q).$$

Proof: By Lemma 5.3.2,

$$\sum_{P\in L_{N,\ell}(\lambda)} q^{\mathrm{dinv}(P)} x^{\mathrm{wt}_+(P)} = \sum_{\substack{I\subseteq[N-1]\\ |I|=\ell}} q^{h_I(\alpha)} N_{\beta/(\alpha+\varepsilon_I)}(X;q)$$

$$\therefore\ \langle z^{N-m}\rangle \sum_{\substack{\lambda\in D_N\\ P\in L_{N,\ell}(\lambda)}} t^{|\delta/\lambda|} \prod_{\substack{1<i\le N\\ c_i(\lambda)=c_{i-1}(\lambda)+1}} (1+zt^{-c_i(\lambda)})\, q^{\mathrm{dinv}(P)} x^{\mathrm{wt}_+(P)} = \langle z^{N-m}\rangle \sum_{\lambda\in D_N} t^{|\delta/\lambda|} \prod_{\substack{1<i\le N\\ c_i(\lambda)=c_{i-1}(\lambda)+1}} (1+zt^{-c_i(\lambda)}) \Big(\sum_{\substack{I\subseteq[N-1]\\ |I|=\ell}} q^{h_I(\alpha)} N_{\beta/(\alpha+\varepsilon_I)}(X;q)\Big)$$

where $\beta=(1, c_2(\lambda)+1, \dots, c_N(\lambda)+1)$, $\alpha=(c_2(\lambda),\dots,c_N(\lambda),0)$ are the LLT data for λ.

Note that a sequence $\vec u=(u_1,u_2,\dots,u_N)\in\mathbb{N}^N$ is a column area sequence of some $\lambda\in D_N$ iff $u_s\le u_{s-1}+1$ $\forall\, 2\le s\le N$ with $u_1=0$. In this case, $|\vec u|=|\delta/\lambda|$

$$\therefore\ \langle z^{N-m}\rangle \sum_{\lambda\in D_N} t^{|\delta/\lambda|} \prod_{\substack{1<i\le N\\ c_i(\lambda)=c_{i-1}(\lambda)+1}} (1+zt^{-c_i(\lambda)}) \Big(\sum_{\substack{I\subseteq[N-1]\\ |I|=\ell}} q^{h_I(\alpha)} N_{\beta/(\alpha+\varepsilon_I)}(X;q)\Big) = \langle z^{N-m}\rangle \sum_{\substack{\vec u\in\mathbb{N}^N\\ u_s\le u_{s-1}+1\ \forall 2\le s\le N\\ u_1=0}} t^{|\vec u|} \prod_{\substack{1<i\le N \text{ s.t.}\\ u_i=u_{i-1}+1}} (1+zt^{-u_i}) \Big(\sum_{\substack{I\subseteq[N-1]\\ |I|=\ell}} q^{h_I(\alpha)} N_{\beta/(\alpha+\varepsilon_I)}(X;q)\Big)$$

$$= \langle z^{N-m}\rangle \sum_{A\subseteq[N]\setminus\{1\}} \sum_{\substack{\vec u\in\mathbb{N}^N\\ \text{s.t. } u_1=0 \text{ and}\\ u_s\le u_{s-1}+1\ \forall 2\le s\le N\\ \text{with } u_i=u_{i-1}+1\ \forall i\in A}} t^{|\vec u|-\sum_{i\in A}u_i} z^{|A|} \Big(\sum_{\substack{I\subseteq[N-1]\\ |I|=\ell}} q^{h_I(\alpha)} N_{\beta/(\alpha+\varepsilon_I)}(X;q)\Big)$$

$J=[N]\setminus A$ with $|A|=N-m$

$$= \sum_{\substack{\{1\}\subseteq J\subseteq[N]\\ |J|=m}} \sum_{u_j=u_{j-1}+1\ \forall j\notin J} t^{\sum_{j\in J}u_j} \sum_{\substack{I\subseteq[N-1]\\ |I|=\ell}} q^{h_I(\alpha)} N_{\beta/(\alpha+\varepsilon_I)}(X;q)$$

we can drop $u_j\le u_{j-1}+1$ b/c $N_{\beta/(\alpha+\varepsilon_I)}=0$ if $\exists j$ s.t. $(\alpha+\varepsilon_I)_j\ge\alpha_j>\beta_j$
Here $\alpha=(u_2,\dots,u_N,0)$ and $\beta=(1,u_2+1,\dots,u_N+1)$.

Note that there is a bijection between $\{\tau\in\mathbb{N}^m : |\tau|=N-m\}$ and $\{J : \{1\}\subseteq J\subseteq[N]\}$:

$$(\tau_1,\tau_2,\dots,\tau_m) \leftrightarrow \{1, \tau_1+2, \tau_1+\tau_2+3, \dots, \overbrace{\tau_1+\dots+\tau_{m-1}}^{\le N-m}+m\}$$

(From $J=\{1<j_1<\dots<j_{m-1}\}$, $j_1-2=\tau_1$, $j_2-j_1-1=\tau_2$, $j_3-j_2-1=\tau_3$, $\dots$, $j_{m-1}-j_{m-2}-1=\tau_{m-1}$, $N-m-\tau_1-\tau_2-\dots-\tau_{m-1}=\tau_m$.)

Then for any fixed J, equivalently τ, $\vec u$ can be written as $u_j=u_{j-1}+1$ $\forall j\notin\{1,\tau_1+2,\tau_1+\tau_2+3,\dots,\tau_1+\dots+\tau_{m-1}+m\}$

$$\vec u=(\underbrace{0,1,2,\dots,\tau_1}_{u_j \text{ with } j<\tau_1+1}, \overset{u_{\tau_1+2}}{a_1}, a_1+1,\dots,a_1+\tau_2, \overset{u_{\tau_1+\tau_2+3}}{a_2},\dots,a_2+\tau_3,\dots, \overset{u_{\tau_1+\dots+\tau_{m-1}+m}}{a_{m-1}},\dots,a_{m-1}+\tau_m) \quad \text{where } (a_1,\dots,a_{m-1})\in\mathbb{N}^{m-1}$$

$$\therefore\ \beta=(1,2,3,\dots,\tau_1+1,a_1+1,a_1+2,\dots,a_1+\tau_2+1,\dots,a_{m-1}+1,\dots,a_{m-1}+\tau_m+1)=\beta_{\vec a\tau}$$

$$\alpha=(1,2,\dots,\tau_1,a_1,a_1+1,\dots,a_1+\tau_2,\dots,a_{m-1},\dots,a_{m-1}+\tau_m,0)=\alpha_{\vec a\tau}$$

$0+a_1+a_2+\dots+a_{m-1}=|\vec a|$

i.e. $$\sum_{\substack{\{1\}\subseteq J\subseteq[N]\\ |J|=m}} \sum_{u_j=u_{j-1}+1\ \forall j\notin J} t^{\sum_{j\in J}u_j} \sum_{\substack{I\subseteq[N-1]\\ |I|=\ell}} q^{h_I(\alpha)} N_{\beta/(\alpha+\varepsilon_I)}(X;q) = \sum_{\substack{I\subseteq[N-1]\\ |I|=\ell}} \sum_{\substack{\tau,(0,\vec a)\in\mathbb{N}^m\\ |\tau|=N-m}} t^{|\vec a|} q^{h_I(\alpha_{\vec a\tau})} N_{\beta_{\vec a\tau}/(\alpha_{\vec a\tau}+\varepsilon_I)}(X;q)$$ and result follows

$\square$

Notation: Given $\vec a\in\mathbb{N}^{m-1}$, $\tau\in\mathbb{N}^m$:

- $j_\uparrow := j+\sum_{x\le j}\tau_x = j+\tau_1+\tau_2+\dots+\tau_j$ $\quad\forall j\in[m]$
- $J_\uparrow := \{j_\uparrow : j\in J\}$ $\quad\forall J\subseteq[m]$

e.g. $m=7$, $\tau=(2,3,1,1,0,2,2)$

$2_\uparrow=2+\tau_1+\tau_2=2+2+3=7$, $5_\uparrow=5+\tau_1+\dots+\tau_5=5+2+3+1+1=12$

$J=\{1,2,5,6\} \Rightarrow J_\uparrow=\{1+\tau_1, 2+\tau_1+\tau_2, 5+\tau_1+\dots+\tau_5, 6+\tau_1+\dots+\tau_6\}=\{3,7,12,15\}$

$[m]_\uparrow=\{3,7,9,11,12,15,18\}$

τ_1+1 $\quad+\tau_2+1$ $\quad+\tau_3+1$ $\quad+\tau_4+1$ $\quad\dots$ $\quad+\tau_7+1$

* $(\beta_{\vec{a}\tau})_{j\uparrow} = a_{j-1} + \tau_j + 1$ (set $a_0 = 0$)

$(\alpha_{\vec{a}\tau})_{j\uparrow} = a_j$ for $1 \le j < m$ and 0 if $j = m$.

* $(\beta_{\vec{a}\tau})_i = (\alpha_{\vec{a}\tau})_i$ if $i \notin [m]_\uparrow$ $\Rightarrow$ row i in $\beta_{\vec{a}\tau}/\alpha_{\vec{a}\tau}$ is empty if $i \notin [m]_\uparrow$

Deleting these empty rows from $\beta_{\vec{a}\tau}/\alpha_{\vec{a}\tau}$ gives $(0,\vec{a}) + \overbrace{(1,\dots,1)}^{m} + \tau \,/\, (\vec{a},0)$ whose row j corresponds to row $j\uparrow$ in $\beta_{\vec{a}\tau}/\alpha_{\vec{a}\tau}$.

e.g. $m=7$, $\vec{a}=(1,3,0,0,1,2)$, $\tau=(2,3,1,1,0,2,2)$, $[m]_\uparrow = \{3,7,9,11,12,15,18\}$

$(0,\vec{a}) + (1,1,1,1,1,1,1) = (\boxed{1},\boxed{2},\boxed{4},\boxed{1},\boxed{1},\boxed{2},\boxed{3})$

$(0,\vec{a}) + \tau + (1,1,1,1,1,1,1) = (\boxed{3},\boxed{5},\boxed{5},\boxed{2},\boxed{1},\boxed{4},\boxed{5})$

$\beta_{\vec{a}\tau} = (1,2,\boxed{3},2,3,4,\boxed{5},4,\boxed{5},1,\boxed{2},\boxed{1},2,3,\boxed{4},3,4,\boxed{5})$

$\alpha_{\vec{a}\tau} = (1,2,\boxed{1},2,3,4,\boxed{3},4,\boxed{0},1,\boxed{0},\boxed{1},2,3,\boxed{2},3,4,\boxed{0})$

$(\beta_{\vec{a}\tau})_i = (\alpha_{\vec{a}\tau})_i$ for $i \notin [7]_\uparrow$

$1\uparrow$ $2\uparrow$ $3\uparrow$ $4\uparrow$ $5\uparrow$ $6\uparrow$ $7\uparrow$

$2 = \tau_1$, $3 = \tau_2$, $1 = \tau_3$, $1 = \tau_4$, $0 = \tau_5$, $2 = \tau_6$, $2 = \tau_7$

Deleting the empty rows, $\beta_{\vec{a}\tau}/\alpha_{\vec{a}\tau}$ becomes $\underbrace{(3,5,5,2,1,4,5)}_{(0,\vec{a})+\tau+\underbrace{(1,\dots,1)}_{7}} / \underbrace{(1,3,0,0,1,2,0)}_{(\vec{a},0)}$

For $J \subseteq [m]$, $\vec{a} \in \mathbb{N}^{m-1}$ and $\tau \in \mathbb{N}^m$, define

$$h'_J(\vec{a},\tau) = |\{(j,r) : j \in J,\ r \in [m],\ j<r,\ a_j \in [a_{r-1}, a_{r-1}+\tau_r - 1]\}| \text{ with } a_0 := 0.$$

e.g. $m=7$, $\vec{a}=(1,3,0,0,1,2)$, $\tau=(2,3,1,1,0,2,2)$, $J=\{1,2,3,5\}$

The intervals $[a_{r-1}, a_{r-1}+\tau_r-1]$ are

	$[0,1]$	$[1,3]$	$[3,3]$	$[0,0]$	$[0,-1] = \phi$	$[1,2]$	$[2,3]$
$a_1 = 1$		✓				✓	
$a_2 = 3$			✓				✓
$a_3 = 0$				✓			
$a_5 = 1$						✓	

$\Rightarrow h'_J(\vec{a},\tau) = |\{(1,2),(1,6),(2,3),(2,7),(3,4),(5,6)\}| = 6$

Claim: $h_{J_\uparrow}(\alpha_{\vec{a}\tau}) = h'_J(\vec{a},\tau) + h_J(\vec{a})$.

e.g. $m=7$, $\vec{a}=(1,3,0,0,1,2)$, $\tau=(2,3,1,1,0,2,2)$, $J=\{1,2,3,5\}$

$J_\uparrow = \{3,7,9,12\}$

$\alpha_{\vec{a}\tau} = (1,2,1,2,3,4,3,4,0,1,0,1,2,3,2,3,4,0)$

$h_{J_\uparrow}(\alpha_{\vec{a}\tau}) = 8$

$h_J(\vec{a}) = 2$ $(1,3,0,0,1,2)$

We showed that $h'_J(\vec{a},\tau) = 6$. Hence $h_{J_\uparrow}(\alpha_{\vec{a}\tau}) = 8 = 6+2 = h'_J(\vec{a},\tau) + h_J(\vec{a})$.

In our example, this is $[18]\setminus\{3,7,9,12\}$.

"unequal positions"

Proof: Recall $\ell(\alpha_{\vec{a}\tau}) = m_\uparrow$. Also, $[m_\uparrow]\setminus J_\uparrow = \underbrace{([m_\uparrow]\setminus[m]_\uparrow)}_{\text{These are the positions where } \alpha_{\vec{a}\tau} \text{ and } \beta_{\vec{a}\tau} \text{ are equal}} \sqcup \underbrace{([m]_\uparrow \setminus J_\uparrow)}_{\text{These are the "unequal" positions that we did not consider}} = ([m_\uparrow]\setminus[m]_\uparrow) \sqcup ([m]\setminus J)_\uparrow$.

$J_\uparrow$ chosen from the "unequal positions" of $\alpha_{\vec{a}\tau}$ & $\beta_{\vec{a}\tau}$, i.e. $J_\uparrow \subseteq [m]_\uparrow$

In our example, $[m]_\uparrow = \{3,7,9,11,12,15,18\}$

in $J_\uparrow$

In our example, $[m]_\uparrow \setminus J_\uparrow = \{11,15,18\} = \{4,6,7\}_\uparrow = ([7]\setminus\underbrace{\{1,2,3,5\}}_{J})_\uparrow$

$\therefore$ We add $\{11,15,18\}$ to $[18]\setminus\{3,7,9,11,12,15,18\}$ which is the same as $[18]\setminus\{3,7,9,12\}$.

Hence, $h_{J_\uparrow}(\alpha_{\vec{a}\tau}) := |\{(i,j) : i \in J_\uparrow,\ j \in [m_\uparrow] \setminus J_\uparrow,\ i<j,\ (\alpha_{\vec{a}\tau})_j = (\alpha_{\vec{a}\tau})_i + 1\}|$

$$= |\underbrace{\{(i,j) : i \in J_\uparrow,\ j \in [m_\uparrow]\setminus[m]_\uparrow,\ i<j,\ (\alpha_{\vec{a}\tau})_j = (\alpha_{\vec{a}\tau})_i + 1\}}_{S_1}| + |\underbrace{\{(i,j) : i \in J_\uparrow,\ j \in ([m]\setminus J)_\uparrow,\ i<j,\ (\alpha_{\vec{a}\tau})_j = (\alpha_{\vec{a}\tau})_i + 1\}}_{S_2}|$$

Note that $(\alpha_{\vec{a}\tau})_{u_\uparrow} = a_u \quad \forall u \in [m-1]$ and $(\alpha_{\vec{a}\tau})_{m_\uparrow} = 0$

$\therefore |S_2| = |\{(i,j) : i \in J_\uparrow,\ j \in ([m]\setminus J)_\uparrow,\ i<j,\ (\alpha_{\vec{a}\tau})_j = (\alpha_{\vec{a}\tau})_i + 1\}|$
$= |\{(u,v) : u \in J,\ v \in [m-1]\setminus J,\ u<v,\ a_v = a_u + 1\}| = h_J(\vec{a})$

a_{u+1}

$a \leq (\alpha_{\vec{a}\tau})_j \leq$

$|S_1| = |\{(i,j) : i \in J_\uparrow,\ j \in [m_\uparrow]\setminus[m]_\uparrow,\ i<j,\ (\alpha_{\vec{a}\tau})_j = (\alpha_{\vec{a}\tau})_i + 1\}|$

$\xrightarrow{i=u_\uparrow\ (u\in J)}$

$= |\{(u,j) : u \in J,\ j \in [m_\uparrow]\setminus[m]_\uparrow,\ u_\uparrow < j,\ (\alpha_{\vec{a}\tau})_j = a_u + 1\}|$

$\because$ j is one of the "equal positions" of $\alpha_{\vec{a}\tau}$ and $\beta_{\vec{a}\tau}$

$r \geq 2$ b/c $u_\uparrow < j$ for some $u \geq 1 \Rightarrow r-1 \geq 1 \Leftrightarrow r \geq 2$

$\therefore$ j is between two "unequal positions", i.e. $\exists!\, r \in [m]$ s.t. $(r-1)_\uparrow < j < r_\uparrow$ $(\Rightarrow u_\uparrow < j < r_\uparrow \Rightarrow u<r)$

Hence $\underset{\overset{\|}{a_{r-1}}}{(\alpha_{\vec{a}\tau})_{(r-1)_\uparrow}} < \underset{\overset{\|}{a_u+1}}{(\alpha_{\vec{a}\tau})_j} \leq \underset{\overset{\|}{a_{r-1}+\tau_r}}{(\alpha_{\vec{a}\tau})_{r_\uparrow - 1}} \Leftrightarrow a_{r-1} < a_u + 1 \leq a_{r-1} + \tau_r \Leftrightarrow a_{r-1}+1 \leq a_u + 1 \leq a_{r-1} + \tau_r \Leftrightarrow a_u \in [a_{r-1}, a_{r-1}+\tau_r - 1]$

$\therefore |S_1| = \{(u,r) : u \in J,\ r \in [m],\ u<r,\ a_u \in [a_{r-1}, a_{r-1}+\tau_r-1]\}| = h_J'(\vec{a}\tau)$.

As a result, $h_J(\alpha_{\vec{a}\tau}) = h_J'(\vec{a}\tau) + h_J(\vec{a})$.

Lemma 5.3.7 : For $J \subseteq [m]$, $\vec{a} \in \mathbb{N}^{m-1}$ and $\tau \in \mathbb{N}^m$, let $I = J_\uparrow$, then

$$N_{\beta_{\vec{a}\tau}/(\alpha_{\vec{a}\tau}+\varepsilon_I)} = q^{\overbrace{d((0,\vec{a})+\tau)}^{\text{cf. §5.1}} - h_J'(\vec{a}\tau)} N_{((0,\vec{a})+(\underbrace{1,\dots,1}_{m})+\tau)/((\vec{a},0)+\varepsilon_J))}.$$

for LHS, these j's are the "unequal positions" of $\beta_{\vec{a}\tau}$ and $\alpha_{\vec{a}\tau}$.
for RHS, these j's are the j^{th} position of $((0,\vec{a})+(1,\dots,1)+\tau)/((\vec{a},0)+\varepsilon_J)$.

Proof: Set $a_0 = 0$. We can assume $a_j + (\varepsilon_J)_j \leq a_{j-1} + \tau_j + 1 \quad \forall j \in [m]$ (otherwise both N's would be 0 by definition).

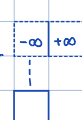
Note that $((0,\vec{a}) + (\underbrace{1,\dots,1}_{m}) + \tau)/((\vec{a},0)+\varepsilon_J)$ is created by removing the empty rows $r \notin [m]_\uparrow$ in $\beta_{\vec{a}\tau}/(\alpha_{\vec{a}\tau}+\varepsilon_I)$. (Note: there may still be some empty rows left if $(\beta_{\vec{a}\tau})_j = (\alpha_{\vec{a}\tau}+\varepsilon_I)_j$ for some $j \in [m]_\uparrow$)

$\therefore$ $N_{\beta_{\vec{a}\tau}/(\alpha_{\vec{a}\tau}+\varepsilon_I)}$ and $N_{((0,\vec{a})+(\underbrace{1,\dots,1}_{m})+\tau)/((\vec{a},0)+\varepsilon_J)}$ shares the same x^{wt} with a different q-power for each term.

Note that the q-power difference is independent of the fillings because the empty row must have labels $S(u) = -\infty$, $S(w) = +\infty$ and hence any value of $S(v)$ with "real box" v must satisfy $S(u) < S(v) < S(w)$.

Hence d counts $\underbrace{\text{w}_0}$-triples of $\beta_{\vec{a}\tau}/(\alpha_{\vec{a}\tau}+\varepsilon_I)$ involving the empty rows, i.e.

(we can drop "increasing" b/c it is automatically increasing)



Note that for $j \notin [m]_\uparrow$, $(\beta_{\vec{a}\tau})_j = (\alpha_{\vec{a}\tau})_j = (\alpha_{\vec{a}\tau}+\varepsilon_I)_j$ (b/c $I = J_\uparrow \subseteq [m]_\uparrow \Rightarrow j \notin I$)

Hence these rows $j \notin [m]_\uparrow$ must be empty and of the form $(b)/(b)$ for some $b \in \{a_{r-1}+1, \dots, a_{r-1}+\tau_r\}$ for some $r \in [m]$ s.t. $(r-1)_\uparrow < j < r_\uparrow$.

Also, there is a "real" box below the left adjacent box of row j, say row $k_\uparrow \in [m]_\uparrow$, of the form $(a_{k-1}+\tau_k+1)/(a_k + (\varepsilon_J)_k)$

$\therefore$ $a_k + (\varepsilon_J)_k < b \leq a_{k-1} + \tau_k + 1$ and $k_\uparrow < j < r_\uparrow \Rightarrow k < r$

Thus $b \in [a_{r-1}+1, a_{r-1}+\tau_r] \cap [a_k + (\varepsilon_J)_k + 1, a_{k-1}+\tau_k+1]$ where $1 \leq k < r \leq m$.

Now $d = \sum_{1\le k<r\le m} \left| [a_{r-1}+1, a_{r-1}+\tau_r] \cap [a_k+(\varepsilon_J)_k+1, a_{k-1}+\tau_k+1] \right|$

$$= \sum_{1\le k<r\le m} \left| [a_{r-1}, a_{r-1}+\tau_r-1] \cap [a_k+(\varepsilon_J)_k, a_{k-1}+\tau_k] \right|$$

$$= \sum_{1\le k<r\le m} \left| [a_k, a_{k-1}+\tau_k] \cap [a_{r-1}, a_{r-1}+\tau_r-1] \right| - \sum_{\substack{1\le k<r\le m\\ k\in J}} \left| \{a_k\} \cap [a_{r-1}, a_{r-1}+\tau_r-1] \right|$$

b/c we assume $a_k \le a_{k-1}+\tau_k+1$

$$= \sum_{1\le k<r\le m} \Big(\big| [\underset{1,\dots,r-1}{a_k}, \infty) \cap [a_{r-1}, a_{r-1}+\tau_r-1] \big| - \big| [a_{k-1}+\tau_k+1, \infty) \cap [a_{r-1}, a_{r-1}+\tau_r-1] \big| \Big) - \underbrace{\left| \{(k,r) : 1\le k<r\le m,\ k\in J,\ a_k \in [a_{r-1}, a_{r-1}+\tau_r-1]\} \right|}_{h'_J(\vec{a},\tau)}$$

$$= \Big(\sum_{1\le k<r\le m} \big| [\underset{0,\dots,r-2}{a_{k-1}}, \infty) \cap [a_{r-1}, a_{r-1}+\tau_r-1] \big| \Big) + \underbrace{\big| [a_{r-1}, \infty) \cap [a_{r-1}, a_{r-1}+\tau_r-1] \big|}_{\left|[a_{r-1}, a_{r-1}+\tau_r-1]\right| = \tau_r} - \underbrace{\big| [\overset{0}{\overset{\shortparallel}{a_0}}, \infty) \cap [a_{r-1}, a_{r-1}+\tau_r-1] \big|}_{\left|[a_{r-1}, a_{r-1}+\tau_r-1]\right| = \tau_r} - \sum_{1\le k<r\le m} \big| [a_{k-1}+\tau_k+1, \infty) \cap [a_{r-1}, a_{r-1}+\tau_r-1] \big| - h'_J(\vec{a},\tau)$$

$$= \sum_{1\le k<r\le m} \Big(\big| [a_{k-1}, \infty) \cap [a_{r-1}, a_{r-1}+\tau_r-1] \big| - \big| [a_{k-1}+\tau_k+1, \infty) \cap [a_{r-1}, a_{r-1}+\tau_r-1] \big| \Big) - h'_J(\vec{a},\tau)$$

$$= \underbrace{\sum_{1\le k<r\le m} \big| [a_{k-1}, a_{k-1}+\tau_k] \cap [a_{r-1}, a_{r-1}+\tau_r-1] \big|}_{d((0,\vec{a}),\tau)} - h'_J(\vec{a},\tau)$$

$$= d((0,\vec{a}),\tau) - h'_J(\vec{a},\tau). \qquad \square$$

Theorem 5.1.1: For $0 \le \ell < m \le N$, we have

$$\underbrace{\langle z^{N-m} \rangle \sum_{\substack{\lambda\in D_N\\ P\in L_{N,\ell}(\lambda)}} t^{|\delta/\lambda|} \prod_{\substack{1< i\le N\\ c_i(\lambda)=c_{i-1}(\lambda)+1}} (1+zt^{-c_i(\lambda)})\, q^{\mathrm{dinv}(P)} x^{\mathrm{wt}_+(P)}}_{\text{c.f. §2}} = \sum_{\substack{J\subseteq[m-1]\\ |J|=\ell}} \sum_{\substack{\tau,(0,\vec{a})\in\mathbb{N}^m\\ |\tau|=N-m}} t^{|\vec{a}|} q^{d((0,\vec{a})+\tau)+h_J(\vec{a})} \underbrace{N_{\beta/\alpha}(X;q)}_{\text{c.f. §5.2}}$$

where $\beta = (0,\vec{a}) + (\underbrace{1,1,\dots,1}_{m}) + \tau$, $\alpha = (\vec{a},0) + \overset{\sum_{j\in J}\varepsilon_j}{\varepsilon_J}$

ε_J = (0,1)-sequence with 1 on the j^{th} pos. iff $j\in J$

Proof: By Lemma 5.3.6, for $0\le\ell<m\le N$,

$$\langle z^{N-m} \rangle \sum_{\substack{\lambda\in D_N\\ P\in L_{N,\ell}(\lambda)}} t^{|\delta/\lambda|} \prod_{\substack{1< i\le N\\ c_i(\lambda)=c_{i-1}(\lambda)+1}} (1+zt^{-c_i(\lambda)})\, q^{\mathrm{dinv}(P)} x^{\mathrm{wt}_+(P)} = \sum_{\substack{I\subseteq[N-1]\\ |I|=\ell}} \sum_{\substack{\tau,(0,\vec{a})\in\mathbb{N}^m\\ |\tau|=N-m}} t^{|\vec{a}|} q^{h_I(\alpha_{\vec{a}\tau})} N_{\beta_{\vec{a}\tau}/(\alpha_{\vec{a}\tau}+\varepsilon_I)}(X;q).$$

If $I \ne J_\uparrow$ for any $J\subseteq[m-1]$, then $(\alpha_{\vec{a}\tau}+\varepsilon_I)_i > (\beta_{\vec{a}\tau})_i$ for some index i (b/c $(\alpha_{\vec{a}\tau})_i = (\beta_{\vec{a}\tau})_i$) and hence $N_{\beta_{\vec{a}\tau}/(\alpha_{\vec{a}\tau}+\varepsilon_I)} = 0$.

Hence we may assume $I = J_\uparrow$ for some $J\subseteq[m-1]$. Then by Lemma 5.3.7,

$$N_{\beta_{\vec{a}\tau}/(\alpha_{\vec{a}\tau}+\varepsilon_I)} = q^{d((0,\vec{a})+\tau)-h'_J(\vec{a},\tau)} N_{\overbrace{((0,\vec{a})+(\underbrace{1,\dots,1}_{m})+\tau)}^{\beta}/\overbrace{((\vec{a},0)+\varepsilon_J)}^{\alpha}}.$$

$$\therefore \sum_{\substack{I\subseteq[N-1]\\ |I|=\ell}} \sum_{\substack{\tau,(0,\vec{a})\in\mathbb{N}^m\\ |\tau|=N-m}} t^{|\vec{a}|} q^{h_I(\alpha_{\vec{a}\tau})} N_{\beta_{\vec{a}\tau}/(\alpha_{\vec{a}\tau}+\varepsilon_I)}(X;q) = \sum_{\substack{J\subseteq[m-1]\\ |J|=\ell}} \sum_{\substack{\tau,(0,\vec{a})\in\mathbb{N}^m\\ |\tau|=N-m}} t^{|\vec{a}|} q^{h_I(\alpha_{\vec{a}\tau}) + d((0,\vec{a})+\tau) - h'_J(\vec{a},\tau)} N_{\beta/\alpha}(X;q)$$

$h_I(\alpha_{\vec{a}\tau}) - h'_J(\vec{a},\tau) = h_J(\vec{a})$ by Claim

$$= \sum_{\substack{J\subseteq[m-1]\\ |J|=\ell}} \sum_{\substack{\tau,(0,\vec{a})\in\mathbb{N}^m\\ |\tau|=N-m}} t^{|\vec{a}|} q^{d((0,\vec{a})+\tau)+h_J(\vec{a})} N_{\beta/\alpha}(X;q). \qquad \square$$