Refer to Path §4 for definitions of $E_\lambda^\sigma(x_1,\dots,x_m;q)$ and $F_\lambda^\sigma(x_1,\dots,x_m;q)$ for $\lambda \in \mathbb{Z}^m$, $\sigma \in S_m$.

Recall the LLT series $\mathcal{L}^\sigma_{\beta/\alpha}(x_1,\dots,x_m;q)$ is defined by:

$$\langle x_\lambda \rangle \mathcal{L}^{\sigma^{-1}}_{\beta/\alpha}(x_1,\dots,x_m;q^{-1}) = \langle E^\sigma_\beta \rangle x_\lambda \cdot E^\sigma_\alpha.$$

Alternatively, by Prop 4.4.2 in Path,

$$\mathcal{L}^\sigma_{\beta/\alpha}(x_1,\dots,x_m;q) = H^m_q\left(\omega_0\left(F^{\sigma^{-1}}_\beta(x_1,\dots,x_m;q)\,\overline{E^{\sigma^{-1}}_\alpha(x_1,\dots,x_m;q)}\right)\right)$$

In this paper, we only need $\sigma=\omega_0$ and $\sigma=\mathrm{id}$.

A semistandard tableau on a tuple of skew row shapes $\nu=\beta/\alpha$ is a map $T:\nu \to [m]$ which is weakly increasing on rows.

Let $SSYT(\nu)$ be the set of all semistandard tableaux of skew row shapes ν.

For $T \in SSYT(\nu)$, $x^{wt(T)} := \prod_{b\in\nu} x_{T(b)}$

Prop 6.2.1 : If $\alpha_i \le \beta_i$ for all $1\le i\le m$, then

$$\mathcal{L}^{\omega_0}_{\beta/\alpha}(x_1,\dots,x_m;q)_{pol} = \sum_{T\in SSYT(\beta/\alpha)} q^{h'_{\omega_0}(T)}\, x^{wt(T)}$$

where $h'_{\omega_0}(T) = \#\ \omega_0$-triples (u,v,w) of β/α s.t. $T(u)\le T(v)\le T(w)$.

Proof: By Corollary 4.5.7 in Path,

$$\mathcal{L}^{\omega_0}_{\beta/\alpha}(x_1,\dots,x_m;q)_{pol} = q^{h_{\omega_0}(\beta/\alpha)}\, \mathcal{G}_{\omega_0(\beta/\alpha)}(x_1,\dots,x_m;q^{-1})$$

$$= q^{h_{\omega_0}(\beta/\alpha)} \cdot \sum_{T\in SSYT(\omega_0(\beta/\alpha))} q^{-\mathrm{inv}(T)} x^T$$

by Remark 4.5.5 in Path

$$= \sum_{T\in SSYT(\omega_0(\beta/\alpha))} q^{h_{\omega_0}(\omega_0^{-1}(T))} x^T$$

by $\omega_0^{-1}=\omega_0$ and $x^T = x^{\omega_0(T)}$

$$= \sum_{T\in SSYT(\omega_0(\beta/\alpha))} q^{h_{\omega_0}(\omega_0(T))} x^{\omega_0(T)}$$

Set $T=\omega_0(S)$ for some $S\in SSYT(\beta/\alpha)$

$$= \sum_{S\in SSYT(\beta/\alpha)} q^{h'_{\omega_0}(S)} x^S$$

where $h'_{\omega_0}(S) = \#\ \omega_0$-triples (u,v,w) of β/α s.t. $S(u)\le S(v)\le S(w)$

change variables $S\leftrightarrow T$

$$= \sum_{T\in SSYT(\beta/\alpha)} q^{h'_{\omega_0}(T)} x^T.$$

$\square$

Prop 6.2.2. (Path Prop. 4.5.2) : For any $\alpha,\beta\in\mathbb{Z}^m$, $\mathcal{L}^{\omega_0}_{\beta/\alpha}(x_1,\dots,x_m;q)_{pol} = (\omega N_{\beta/\alpha})(x_1,\dots,x_m;q)$.

By putting $\sigma=\omega_0$ and the fact that $N_{\beta/\alpha}$ in this paper is the same as $N^{\omega_0}_{\beta/\alpha}$ in Path.