Lemma 6.3.1: For $\vec{a} \in \mathbb{N}^{m-1}$, $\omega_0 \in S_m$ and $\widetilde{\omega_0} \in S_{m-1}$ the permutations of maximum length, we have

$$E^{\omega_0}_{(\vec{a},0)}(x_1,\dots,x_m;q) = E^{\widetilde{\omega_0}}_{\vec{a}}(x_1,\dots,x_{m-1};q)$$

$$F^{\omega_0}_{(0,\vec{a})}(x_1,\dots,x_m;q) = F^{\widetilde{\omega_0}}_{\vec{a}}(x_2,\dots,x_m;q).$$

Proof: By Lemma 4.3.4 in Path,

$$E^{\omega_0}_{(\vec{a},0)}(x_1,\dots,x_m;q) = E^{\widetilde{\omega_0}}_{\vec{a}}(x_1,\dots,x_{m-1};q)\overbrace{E^{id}_{(0)}(x_m;q)}^{1} = E^{\widetilde{\omega_0}}_{\vec{a}}(x_1,\dots,x_{m-1};q)$$

$$E^{id}_{(0,-\vec{a})}(x_1,\dots,x_m;q) = \underbrace{E^{id}_{(0)}(x_1;q)}_{1} E^{id}_{-\vec{a}}(x_2,\dots,x_m;q)$$

$$\therefore F^{\omega_0}_{(0,\vec{a})}(x_1,\dots,x_m;q) = \overline{E^{\omega_0\omega_0}_{(0,-\vec{a})}(x_1,\dots,x_m;q)} = \overline{E^{id}_{(0,-\vec{a})}(x_1,\dots,x_m;q)} = \overline{E^{id}_{-\vec{a}}(x_2,\dots,x_m;q)} = F^{\omega_0}_{\vec{a}}(x_2,\dots,x_m;q).$$

$\square$

Lemma 6.3.2: For $\ell \le m$, $\vec{a} \in \mathbb{Z}^m$, we have

$$\overline{e_\ell(x_1,\dots,x_m)}\ \overline{E^{\omega_0}_{\vec{a}}(x_1,\dots,x_m;q)} = \sum_{\substack{I\subseteq[m]\\|I|=\ell}} q^{h_I(\vec{a})}\,\overline{E^{\omega_0}_{\vec{a}+\varepsilon_I}(x_1,\dots,x_m;q)}$$

where $h_I(\vec{a}) = |\{(i,j) : 1 \le i < j \le m,\ \vec{a}_j = \vec{a}_i + 1,\ i \in I,\ j \notin I\}|$.

Proof: By Lemma 4.5.1 in Path,

$$e_\ell(x_1,\dots,x_m) E^{\omega_0}_{\vec{a}}(x_1,\dots,x_m;q) = \sum_{\substack{I\subseteq[m]\\|I|=\ell}} q^{-h_I} E^{\omega_0}_{\vec{a}+\varepsilon_I}(x_1,\dots,x_m;q)$$

where $h_I = |\{(i,j): 1\le i<j\le m,\ i\in I,\ j\notin I,\ \vec{a}_i = \vec{a}_j \text{ if } \underbrace{\omega_0(i)<\omega_0(j)}_{\text{impossible}}, \text{ or } \vec{a}_j = \vec{a}_i+1 \text{ if } \underbrace{\omega_0(i)>\omega_0(j)}_{\text{always true}}\}|$

$$= |\{(i,j): 1\le i<j\le m,\ i\in I,\ j\notin I,\ \vec{a}_j = \vec{a}_i + 1\}| = h_I(\vec{a})$$

$$\therefore\ e_\ell(x_1,\dots,x_m) E^{\omega_0}_{\vec{a}}(x_1,\dots,x_m;q) = \sum_{\substack{I\subseteq[m]\\|I|=\ell}} q^{-h_I(\vec{a})} E^{\omega_0}_{\vec{a}+\varepsilon_I}(x_1,\dots,x_m;q)$$

Hence
$$\overline{e_\ell(x_1,\dots,x_m) E^{\omega_0}_{\vec{a}}(x_1,\dots,x_m;q)} = \sum_{\substack{I\subseteq[m]\\|I|=\ell}} q^{h_I(\vec{a})}\,\overline{E^{\omega_0}_{\vec{a}+\varepsilon_I}(x_1,\dots,x_m;q)}.$$

$\square$

Lemma 6.3.3: For $\lambda \in \mathbb{Z}^m$ and $\sigma \in S_m$, we have

$$F^{\sigma}_{\lambda}(x_1,\dots,x_m;q) = \omega_0 E^{\omega_0\sigma}_{\omega_0\lambda}(x_1,\dots,x_m;q^{-1}).$$

Proof: First note that $\omega_0 s_i \omega_0 = s_{m-i}$. If $\omega = s_{i_1}s_{i_2}\cdots s_{i_r}$ is reduced, then $\omega_0\omega\omega_0 = s_{m-i_1}s_{m-i_2}\cdots s_{m-i_r}$ is also reduced (b/c $\ell(\omega)=r$ and $\ell(\omega) = \underset{\ell(\omega_0\omega\omega_0)}{\underset{\shortparallel}{\ell(\omega_0\omega\omega_0^{-1})}}$)

Hence $T_{\omega_0\omega\omega_0} = T_{m-i_1}T_{m-i_2}\cdots T_{m-i_r}$

Define $T_i' = \omega_0 T_{m-i}\,\omega_0$

$$= \omega_0\left(q s_{m-i} + (1-q)\frac{1}{1-\frac{x_{m-i+1}}{x_{m-i}}}(s_{m-i}-1)\right)\omega_0$$

$$= q s_i + (1-q)\frac{1}{1-\frac{x_i}{x_{i+1}}}(s_i - 1)$$

Hence $T_i' = \overline{T_i(x;q^{-1})} = T_i(x;q^{-1})^{-1}$, the conjugate of T_i by $x^\lambda \mapsto x^{-\lambda}$

Suppose $w \in S_m$ s.t. $\lambda = w(\lambda_+)$. Then $-w_0(\lambda_+) = (-\lambda)_+$ and $w_0 w w_0((-\lambda)_+) = w_0 w(-(\lambda_+)) = -w_0(\lambda)$.

Let $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ be a reduced decomposition of w, and hence $w_0 w w_0 = s_{m-i_1} s_{m-i_2} \cdots s_{m-i_r}$. Using the recurrence of E_λ^σ,

we have $$E_{-w_0(\lambda)}^{w_0 \sigma w_0} = q^{k_1} T_{m-i_1}^{a_1} E_{s_{m-i_1}(-w_0(\lambda))}^{s_{m-i_1}(w_0\sigma w_0)} \quad \text{where} \begin{cases} k_1 = -\mathbb{1}\left((-w_0(\lambda))_{m-i_1} \leq (-w_0(\lambda))_{m-i_1+1}\right),\ a_1 = 1 & \text{if } s_{m-i_1}(w_0\sigma w_0) > w_0\sigma w_0 \\ k_1 = \mathbb{1}\left((-w_0(\lambda))_{m-i_1} \geq (-w_0(\lambda))_{m-i_1+1}\right),\ a_1 = -1 & \text{if } s_{m-i_1}(w_0\sigma w_0) < w_0\sigma w_0 \end{cases}$$

$$= q^{k_1} T_{m-i_1}^{a_1} E_{-w_0 s_{i_1}(\lambda)}^{w_0(s_{i_1}\sigma)w_0}$$

$\Updownarrow$

equivalently, $$\begin{cases} k_1 = -\mathbb{1}(-\lambda_{i_1+1} \leq -\lambda_{i_1}),\ a_1 = 1 & \text{if } w_0 s_{i_1}\sigma w_0 > w_0\sigma w_0 \\ k_1 = \mathbb{1}(-\lambda_{i_1+1} \geq -\lambda_{i_1}),\ a_1 = -1 & \text{if } w_0 s_{i_1}\sigma w_0 < w_0\sigma w_0 \end{cases}$$

$\Updownarrow$

$$\begin{cases} k_1 = -\mathbb{1}(\lambda_{i_1} \leq \lambda_{i_1+1}),\ a_1 = 1 & \text{if } s_{i_1}\sigma > \sigma \\ k_1 = \mathbb{1}(\lambda_{i_1} \geq \lambda_{i_1+1}),\ a_1 = -1 & \text{if } s_{i_1}\sigma < \sigma \end{cases}$$

$$E_{-w_0(\lambda)}^{w_0\sigma w_0} = q^{k_1} \cdot q^{k_2} T_{m-i_1}^{a_1} T_{m-i_2}^{a_2} E_{s_{m-i_2}(-w_0 s_{i_1}(\lambda))}^{s_{m-i_2}(w_0 s_{i_1}\sigma w_0)} \quad \text{where} \begin{cases} k_2 = -\mathbb{1}((s_{i_1}\lambda)_{i_2} \leq (s_{i_1}\lambda)_{i_2+1}),\ a_2 = 1 & \text{if } s_{i_2}s_{i_1}\sigma > s_{i_1}\sigma \\ k_2 = \mathbb{1}((s_{i_1}\lambda)_{i_2} \geq (s_{i_1}\lambda)_{i_2+1}),\ a_2 = -1 & \text{if } s_{i_2}s_{i_1}\sigma < s_{i_1}\sigma \end{cases}$$

$$= q^{k_1+k_2} T_{m-i_1}^{a_1} T_{m-i_2}^{a_2} E_{-w_0 s_{i_2} s_{i_1}(\lambda)}^{w_0 s_{i_2} s_{i_1}\sigma w_0}$$

Applying the recurrence using $s_{i_3}, \ldots, s_{i_r}$, we have

$$E_{-w_0(\lambda)}^{w_0\sigma w_0} = q^{k_1+k_2+\cdots+k_r} T_{m-i_1}^{a_1} T_{m-i_2}^{a_2} \cdots T_{m-i_r}^{a_r} E_{-w_0 s_{i_r} s_{i_{r-1}} \cdots s_{i_2} s_{i_1}(\lambda)}^{w_0 s_{i_r} s_{i_{r-1}} \cdots s_{i_1}\sigma w_0}$$

$$= q^{k_1+\cdots+k_r} T_{m-i_1}^{a_1} \cdots T_{m-i_r}^{a_r} E_{\boxed{-w_0(\lambda_+)}}^{w_0 w^{-1}\sigma w_0} \quad \text{dominant, } = (-\lambda)_+$$

$$= q^{k_1+\cdots+k_r} T_{m-i_1}^{a_1} \cdots T_{m-i_r}^{a_r} x^{(-\lambda)_+} \quad \text{where} \begin{cases} k_j = -\mathbb{1}((s_{i_{j-1}} \cdots s_{i_1}\lambda)_{i_j} \leq (s_{i_{j-1}} \cdots s_{i_1}\lambda)_{i_j+1}),\ a_j = 1 & \text{if } s_{i_j} \cdots s_{i_1}\sigma > s_{i_{j-1}} \cdots s_{i_1}\sigma \\ k_j = \mathbb{1}((s_{i_{j-1}} \cdots s_{i_1}\lambda)_{i_j} \geq (s_{i_{j-1}} \cdots s_{i_1}\lambda)_{i_j+1}),\ a_j = -1 & \text{if } s_{i_j} \cdots s_{i_1}\sigma < s_{i_{j-1}} \cdots s_{i_1}\sigma \end{cases}$$

$$= q^{k_1+\cdots+k_r} T_{m-i_1}^{a_1} \cdots T_{m-i_r}^{a_r} w_0 x^{-(\lambda_+)} \quad \text{for all } 1 \leq j \leq r.$$

$$E_\lambda^\sigma(x_1^{-1}, x_2^{-1}, \ldots, x_m^{-1}; q) = \overline{E_\lambda^\sigma(x_1, \ldots, x_m; q^{-1})} = \overline{q^{-p_1} \overset{T_{i_1}(x_1,\ldots,x_m;q^{-1})}{T_{i_1}^{b_1}} E_{s_{i_1}\lambda}^{s_{i_1}\sigma}(x_1, \ldots, x_m; q^{-1})} \quad \text{where} \begin{cases} p_1 = -\mathbb{1}(\lambda_{i_1} \leq \lambda_{i_1+1}),\ b_1 = 1 & \text{if } s_{i_1}\sigma > \sigma \\ p_1 = \mathbb{1}(\lambda_{i_1} \geq \lambda_{i_1+1}),\ b_1 = -1 & \text{if } s_{i_1}\sigma < \sigma \end{cases}$$

Hence $p_1 = k_1$, $b_1 = a_1$

$$= \overline{q^{-k_1} T_{i_1}^{a_1} E_{s_{i_1}\lambda}^{s_{i_1}\sigma}(x_1, \ldots, x_m; q^{-1})}$$

$$= q^{k_1} (T_{i_1}')^{a_1} \overline{E_{s_{i_1}\lambda}^{s_{i_1}\sigma}(x_1, \ldots, x_m; q^{-1})}$$

$$= q^{k_1} (T_{i_1}')^{a_1} \cdot q^{p_2} (T_{i_2}')^{b_2} \overline{E_{s_{i_2}s_{i_1}\lambda}^{s_{i_2}s_{i_1}\sigma}(x_1, \ldots, x_m; q^{-1})} \quad \text{where} \begin{cases} p_2 = -\mathbb{1}((s_{i_1}\lambda)_{i_2} \leq (s_{i_1}\lambda)_{i_2}),\ b_2 = 1 & \text{if } s_{i_1}\sigma > \sigma \\ p_2 = \mathbb{1}((s_{i_1}\lambda)_{i_2} \geq (s_{i_1}\lambda)_{i_2}),\ b_2 = -1 & \text{if } s_{i_1}\sigma < \sigma \end{cases}$$

Hence $p_2 = k_2$, $b_2 = a_2$

$$= q^{k_1+k_2} (T_{i_1}')^{a_1} (T_{i_2}')^{a_2} \overline{E_{s_{i_2}s_{i_1}\lambda}^{s_{i_2}s_{i_1}\sigma}(x_1, \ldots, x_m; q^{-1})}$$

by a similar argument $\rightarrow$ $\vdots$

$$= q^{k_1+\cdots+k_r} (T_{i_1}')^{a_1} \cdots (T_{i_r}')^{a_r} \overline{E_{s_{i_r} \cdots s_{i_1}\lambda}^{s_{i_r} \cdots s_{i_1}\sigma}(x_1, \ldots, x_m; q^{-1})}$$

$$= q^{k_1+\cdots+k_r} (T_{i_1}')^{a_1} \cdots (T_{i_r}')^{a_r} \overline{E_{\lambda_+}^{w^{-1}\sigma}(x_1, \ldots, x_m; q^{-1})}$$

$$= q^{k_1+\cdots+k_r} (w_0 T_{m-i_1}^{a_1} w_0) \cdots (w_0 T_{m-i_r}^{a_r} w_0) \overline{x^{\lambda_+}}$$

$$= q^{k_1+\cdots+k_r} w_0 T_{m-i_1}^{a_1} \cdots T_{m-i_r}^{a_r} w_0 x^{-\lambda_+} = w_0 E_{-w_0(\lambda)}^{w_0\sigma w_0}$$

$\therefore$ By substituting $\sigma \mapsto \sigma w_0$, $\lambda \mapsto -\lambda$, $q \mapsto q^{-1}$,

$$\underbrace{E^{\sigma w_0}_{-\lambda}(x_1^{-1},\dots,x_m^{-1};q^{-1})}_{\overline{E^{\sigma w_0}_{-\lambda}(x_1,\dots,x_m;q)}} = w_0\, E^{w_0\sigma}_{w_0(\lambda)}(x_1,\dots,x_m;q^{-1})$$

i.e. $$F^{\sigma}_{\lambda}(x_1,\dots,x_m;q) = w_0\, E^{w_0\sigma}_{w_0(\lambda)}(x_1,\dots,x_m;q^{-1}).$$

$\square$

Lemma 6.3.4: Let $\alpha,\beta \in \mathbb{Z}^m$ and $f \in \Bbbk[x_1^{\pm1},\dots,x_m^{\pm1}]^{S_m}$. For any $\sigma \in S_m$, we have

$$\langle E^{w_0\sigma w_0}_{w_0\beta}(x_1,\dots,x_m;q^{-1})\rangle f(x)\cdot E^{w_0\sigma w_0^{-1}}_{w_0\alpha}(x_1,\dots,x_m;q^{-1}) = \langle F^{\sigma}_{-\alpha}(x_1,\dots,x_m;q)\rangle f(x)\cdot F^{\sigma}_{-\beta}(x_1,\dots,x_m;q).$$

Proof: We will prove a stronger statement that for any $f \in \Bbbk[x_1^{\pm1},\dots,x_m^{\pm1}]$,

$$\langle E^{w_0\sigma w_0}_{w_0\beta}(x_1,\dots,x_m;q^{-1})\rangle f(x)\cdot E^{w_0\sigma w_0^{-1}}_{w_0\alpha}(x_1,\dots,x_m;q^{-1}) = \langle F^{\sigma}_{-\alpha}(x_1,\dots,x_m;q)\rangle (w_0 f(x))\cdot F^{\sigma}_{-\beta}(x_1,\dots,x_m;q) \quad (*)$$

Then result follows because $f = w_0 f$ when f is symmetric.

By Lemma 6.3.3,

$$\langle F^{\sigma}_{-\alpha}(x_1,\dots,x_m;q)\rangle\, w_0 f(x)\cdot F^{\sigma}_{-\beta}(x_1,\dots,x_m;q)$$

$$= \langle w_0 E^{w_0\sigma}_{-w_0\alpha}(x_1,\dots,x_m;q^{-1})\rangle\, w_0 f(x)\cdot w_0 E^{w_0\sigma}_{-w_0\beta}(x_1,\dots,x_m;q^{-1})$$

$$= \langle E^{w_0\sigma}_{-w_0\alpha}(x_1,\dots,x_m;q^{-1})\rangle f(x)\, E^{w_0\sigma}_{-w_0\beta}(x_1,\dots,x_m;q^{-1})$$

By Prop. 4.3.2 in Path, $\{E^{\sigma}_{\lambda}(x_1,\dots,x_m;q)\}_\lambda$ and $\{\overline{F^{\sigma}_{\lambda}(x_1,\dots,x_m;q)}\}_\lambda = \{E^{\sigma w_0}_{-\lambda}(x_1,\dots,x_m;q)\}$ are dual bases w.r.t. $\langle\,,\rangle_q$ where

$$\langle f,g\rangle_q := \langle x^0\rangle fg \prod_{i<j} \frac{1-\frac{x_i}{x_j}}{1-q^{-1}\frac{x_i}{x_j}}.$$

$$\therefore \langle E^{w_0\sigma}_{-w_0\alpha}(x_1,\dots,x_m;q^{-1})\rangle f(x)\, E^{w_0\sigma}_{-w_0\beta}(x_1,\dots,x_m;q^{-1}) = \langle E^{w_0\sigma w_0}_{-(-w_0\alpha)}(x_1,\dots,x_m;q^{-1}),\, f(x)\, E^{w_0\sigma}_{-w_0\beta}(x_1,\dots,x_m;q^{-1})\rangle_{q^{-1}}$$

RHS of (*)

$$= \langle x^0\rangle\, E^{w_0\sigma w_0}_{w_0\alpha}(x_1,\dots,x_m;q^{-1})\cdot f(x)\cdot E^{w_0\sigma}_{-w_0\beta}(x_1,\dots,x_m;q^{-1}) \prod_{1\le i<j\le m} \frac{1-\frac{x_i}{x_j}}{1-q\frac{x_i}{x_j}}$$

$$= \langle f(x)\, E^{w_0\sigma w_0}_{w_0\alpha}(x_1,\dots,x_m;q^{-1}),\, E^{w_0\sigma}_{-w_0\beta}(x_1,\dots,x_m;q^{-1})\rangle_{q^{-1}}$$

LHS of (*)

$$= \langle E^{w_0\sigma w_0}_{w_0\beta}(x_1,\dots,x_m;q^{-1})\rangle f(x)\, E^{w_0\sigma w_0}_{w_0\alpha}(x_1,\dots,x_m;q^{-1}).$$

$\square$

Lemma 6.3.5: For w_0 the maximal permutation in S_m and $\eta \in \mathbb{N}^m$, we have

$$h_\ell(x_1,\dots,x_m)\, F^{w_0}_{\eta}(x_1,\dots,x_m;q) = \sum_{\substack{\tau\in\mathbb{N}^m\\ |\tau|=\ell}} q^{d(\eta,\tau)}\, F^{w_0}_{\eta+\tau}(x_1,\dots,x_m;q)$$

where $d(\eta;\tau) := \sum_{1\le j<r\le m} \left| [\eta_j;\eta_j+\tau_j] \cap [\eta_r,\eta_r+\tau_r-1] \right|$

Proof: $\langle h_\ell(x_1,\dots,x_m)\rangle\, \mathcal{L}^{w_0}_{w_0(\beta/\alpha)}(x_1,\dots,x_m;q) = \langle x_{(\ell,0,\dots,0)}\rangle\, \mathcal{L}^{w_0}_{w_0(\beta/\alpha)}(x_1,\dots,x_m;q) = \langle E^{w_0}_{w_0(\beta)}(x_1,\dots,x_m;q^{-1})\rangle\, h_\ell(x_1,\dots,x_m)\cdot E^{w_0}_{w_0(\alpha)}(x_1,\dots,x_m;q^{-1})$

by Lemma 6.3.4 with $\sigma = w_0$

$$= \langle F^{w_0}_{-\alpha}(x_1,\dots,x_m;q)\rangle\, h_\ell(x_1,\dots,x_m)\, F^{w_0}_{-\beta}(x_1,\dots,x_m;q)$$

By Prop. 6.2.1,

$$\mathcal{L}^{w_0}_{w_0(\beta/\alpha)}(x_1,\dots,x_m;q)_{pol} = \sum_{T\in SSYT(w_0(\beta/\alpha))} q^{h'_{w_0}(T)} x^{wt(T)}$$ where $h'_{w_0}(T) = \#$ w_0-triples (u,v,w) of $w_0(\beta/\alpha)$ s.t. $T(u)\le T(v)\le T(w)$

Specializing all but one variable to zero, say $x_i\neq 0$ for some i and $x_j=0$ $\forall j\neq i$, then

all boxes are filled with the letter i

$$\mathcal{L}^{w_0}_{w_0(\beta/\alpha)}(0,0,\dots,0,x_i,0,\dots,0;q)_{pol} = \sum_{T\in SSYT(w_0(\beta/\alpha))} q^{h'_{w_0}(T)} x_i^{|w_0(\beta/\alpha)|}$$

where $h'_{w_0}(T) = \#$ w_0-triples of $w_0(\beta/\alpha)$ (b/c $T(u)=T(v)=T(w)=i$)
$= h_{w_0}(w_0(\beta/\alpha))$

$$\therefore \langle h_\ell(x_1,\dots,x_m)\rangle\, \mathcal{L}^{w_0}_{w_0(\beta/\alpha)}(x_1,\dots,x_m;q) = \begin{cases} \sum_{T\in SSYT(w_0(\beta/\alpha))} q^{h'_{w_0}(T)} & \text{if } |\beta/\alpha|=\ell \\ 0 & \text{otherwise}\end{cases}$$

$$= \sum_{j<r} \left| [\alpha_r+1,\beta_r]\cap[\alpha_j,\beta_j]\right|$$

b/c $\underline{w_0}(\beta/\alpha)$ instead of β/α

$$\therefore \langle F^{w_0}_{-\alpha}(x_1,\dots,x_m;q)\rangle\, h_\ell(x_1,\dots,x_m)\, F^{w_0}_{-\beta}(x_1,\dots,x_m;q) = \begin{cases} \sum_{T\in SSYT(w_0(\beta/\alpha))} q^{h'_{w_0}(T)} & \text{if } |\beta/\alpha|=\ell \\ 0 & \text{otherwise}\end{cases}$$

i.e. $$h_\ell(x_1,\dots,x_m)\, F^{w_0}_{\beta}(x_1,\dots,x_m;q) = \sum_{\substack{T\in SSYT(w_0(\beta/\alpha))\\ |\beta/\alpha|=\ell}} q^{h'_{w_0}(T)} F^{w_0}_{\alpha}(x_1,\dots,x_m;q)$$

Put $\beta=-\eta$, $\alpha=-\eta-\tau$, we have

$$h'_{w_0}(T) = \sum_{j<r} \left| [-\eta_r-\tau_r+1,\ -\eta_r] \cap [-\eta_j-\tau_j,\ -\eta_j] \right|$$

$$= \sum_{j<r} \left| [\eta_r,\ \eta_r+\tau_r-1] \cap [\eta_j,\ \eta_j+\tau_j] \right| = d(\eta,\tau)$$

and hence $$h_\ell(x_1,\dots,x_m)\, F^{w_0}_{\eta}(x_1,\dots,x_m;q) = \sum_{\substack{\tau\in\mathbb{N}^m\\ |\tau|=\ell}} q^{d(\eta,\tau)} F^{w_0}_{\eta+\tau}(x_1,\dots,x_m;q).$$

- $\beta/\alpha = -\eta/\eta-\tau \cong \tau\in\mathbb{N}^m$ b/c $\beta,\alpha\in\mathbb{N}^m$.
- $|\beta/\alpha|=\ell \Rightarrow |\tau|=\ell$

Theorem 6.3.6: For $0\le \ell< m\le N$, $w_0\in S_m$ the maximum length permutation, we have

$$\frac{\prod_{i+1<j\le m}\left(1-qt\frac{x_i}{x_j}\right)}{\prod_{1\le i<j\le m}\left(1-t\frac{x_i}{x_j}\right)}\, x_1x_2\cdots x_m\, h_{N-m}(x_1,\dots,x_m)\,\overline{e_\ell(x_2,\dots,x_m)} = \sum_{\substack{(0,\vec{a}),\tau\in\mathbb{N}^m\\ |\tau|=N-m}} \sum_{\substack{I\subseteq[m-1]\\ |I|=\ell}} t^{|\vec{a}|} q^{d((0,\vec{a}),\tau)+h_I(\vec{a})}\, w_0\left(F^{w_0}_{(0,\vec{a})+\tau+(1^m)}(x_1,\dots,x_m;q)\,\overline{E^{w_0}_{(\vec{a},0)+\varepsilon_I}(x_1,\dots,x_m;q)}\right)$$

Proof: By Theorem 5.1.1 in Path, we have

$$\frac{\prod_{1\le i<j\le m-1}(1-qt\,x_iy_j)}{\prod_{1\le i\le j\le m-1}(1-t\,x_iy_j)} = \sum_{\vec{a}\in\mathbb{N}^{m-1}} t^{|\vec{a}|} E^{\tilde{\sigma}}_{\vec{a}}(x_1,\dots,x_{m-1};q^{-1})\, F^{\tilde{\sigma}}_{\vec{a}}(y_1,\dots,y_{m-1};q) \quad \forall\, \tilde{\sigma}\in S_{m-1}.$$

Take $\tilde{\sigma}=\tilde{w}_0$ the maximum length permutation in S_{m-1}, replace x_i by x_i^{-1} and then $y_j=x_{j+1}$, we have

$$\frac{\prod_{1\le i<j\le m-1}\left(1-qt\frac{x_{j+1}}{x_i}\right)}{\prod_{1\le i\le j\le m-1}\left(1-t\frac{x_{j+1}}{x_i}\right)} = \sum_{\vec{a}\in\mathbb{N}^{m-1}} t^{|\vec{a}|} E^{\tilde{w}_0}_{\vec{a}}(x_1^{-1},\dots,x_{m-1}^{-1};q^{-1})\, F^{\tilde{w}_0}_{\vec{a}}(x_2,\dots,x_m;q)$$

$j\le m-1$ in the previous step $\Rightarrow \underbrace{j+1}_{\text{new } j \text{ here}}\le m$

i.e. $$\frac{\prod_{\substack{1\le i,j\le m\\ i+1<j}}\left(1-qt\frac{x_j}{x_i}\right)}{\prod_{1\le i<j\le m}\left(1-t\frac{x_j}{x_i}\right)} = \sum_{\vec{a}\in\mathbb{N}^{m-1}} t^{|\vec{a}|}\, \overline{E^{\tilde{w}_0}_{\vec{a}}(x_1,\dots,x_{m-1};q)}\, F^{\tilde{w}_0}_{\vec{a}}(x_2,\dots,x_m;q).$$

$\because\ x_1\cdots x_m F^{\tilde{\omega}_0}_{\vec{a}}(x_2,\dots,x_m;q) = x_1\cdots x_m F^{\omega_0}_{(0,\vec{a})}(x_1,\dots,x_m;q)$ (by Lemma 6.3.1)

by def.

$$= F^{\omega_0}_{(0,\vec{a})+\underbrace{(1,1,\dots,1)}_{m}}(x_1,\dots,x_m;q)$$

$F^{\omega_0}_{(0,\vec{a})+\underbrace{(1,\dots,1)}_{m}}(x_1,\dots,x_m;q)$

$$\therefore \frac{\prod_{\substack{1\le i\le j\le m\\ i+1<j}}\left(1-qt\frac{x_j}{x_i}\right)}{\prod_{1\le i<j\le m}\left(1-t\frac{x_j}{x_i}\right)}(x_1\cdots x_m)h_{N-m}(x_1,\dots,x_m) = \sum_{\vec{a}\in\mathbb{N}^{m-1}} t^{|\vec{a}|}(x_1\cdots x_m)h_{N-m}(x_1,\dots,x_m)\cdot F^{\tilde{\omega}_0}_{\vec{a}}(x_2,\dots,x_m;q)\overline{E^{\tilde{\omega}_0}_{\vec{a}}(x_1,\dots,x_{m-1};q)}$$

$$= \sum_{\vec{a}\in\mathbb{N}^{m-1}} t^{|\vec{a}|} h_{N-m}(x_1,\dots,x_m) F^{\omega_0}_{(0,\vec{a})+\underbrace{(1,\dots,1)}_{m}}(x_1,\dots,x_m;q)\overline{E^{\tilde{\omega}_0}_{\vec{a}}(x_1,\dots,x_{m-1};q)}$$

by Lemma 6.3.5

$$= \sum_{\vec{a}\in\mathbb{N}^{m-1}} t^{|\vec{a}|}\sum_{\substack{\tau\in\mathbb{N}^m\\ |\tau|=N-m}} q^{d((0,\vec{a})+\underbrace{(1,1,\dots,1)}_{m},\tau)} F^{\omega_0}_{(0,\vec{a})+(1,1,\dots,1)+\tau}(x_1,\dots,x_m;q)\overline{E^{\tilde{\omega}_0}_{\vec{a}}(x_1,\dots,x_{m-1};q)}$$

Note that $d(\eta+\underbrace{(1,1,\dots,1)}_{m},\tau)=d(\eta,\tau)\quad\forall\,\eta,\tau\in\mathbb{Z}^m$ b/c

$$d(\eta+\underbrace{(1,\dots,1)}_{m},\tau) = \sum_{1\le j<r\le m}\left|[\eta_j+1,\eta_j+1+\tau_j]\cap[\eta_r+1,\eta_r+\tau_r]\right|$$

$$= \sum_{1\le j<r\le m}\left|[\eta_j,\eta_j+\tau_j]\cap[\eta_j,\eta_r+\tau_r-1]\right|$$

$$= d(\eta,\tau)$$

$$\therefore \frac{\prod_{\substack{1\le i\le j\le m\\ i+1<j}}\left(1-qt\frac{x_j}{x_i}\right)}{\prod_{1\le i<j\le m}\left(1-t\frac{x_j}{x_i}\right)}(x_1\cdots x_m)h_{N-m}(x_1,\dots,x_m) = \sum_{\substack{(0,\vec{a}),\tau\in\mathbb{N}^m\\ |\tau|=N-m}} t^{|\vec{a}|} q^{d((0,\vec{a}),\tau)} F^{\omega_0}_{(0,\vec{a})+(1,1,\dots,1)+\tau}(x_1,\dots,x_m;q)\overline{E^{\tilde{\omega}_0}_{\vec{a}}(x_1,\dots,x_{m-1};q)}$$

$$\therefore \frac{\prod_{\substack{1\le i\le j\le m\\ i+1<j}}\left(1-qt\frac{x_j}{x_i}\right)}{\prod_{1\le i<j\le m}\left(1-t\frac{x_j}{x_i}\right)}(x_1\cdots x_m)h_{N-m}(x_1,\dots,x_m)\overline{e_\ell(x_1,\dots,x_{m-1})} = \sum_{\substack{(0,\vec{a}),\tau\in\mathbb{N}^m\\ |\tau|=N-m}} t^{|\vec{a}|} q^{d((0,\vec{a}),\tau)} F^{\omega_0}_{(0,\vec{a})+(1,1,\dots,1)+\tau}(x_1,\dots,x_m;q)\underbrace{\overline{E^{\tilde{\omega}_0}_{\vec{a}}(x_1,\dots,x_{m-1};q)}\,\overline{e_\ell(x_1,\dots,x_{m-1})}}$$

By Lemma 6.3.2, this is $\sum_{\substack{I\subseteq[m-1]\\ |I|=\ell}} q^{h_I(\vec{a})}\overline{E^{\tilde{\omega}_0}_{\vec{a}+\varepsilon_I}(x_1,\dots,x_{m-1};q)}$

$$= \sum_{\substack{(0,\vec{a}),\tau\in\mathbb{N}^m\\ |\tau|=N-m}}\ \sum_{\substack{I\subseteq[m-1]\\ |I|=\ell}} t^{|\vec{a}|} q^{d((0,\vec{a}),\tau)+h_I(\vec{a})} F^{\omega_0}_{(0,\vec{a})+\underbrace{(1,1,\dots,1)}_{m}+\tau}(x_1,\dots,x_m;q)\underbrace{\overline{E^{\tilde{\omega}_0}_{\vec{a}+\varepsilon_I}(x_1,\dots,x_{m-1};q)}}_{\overline{E^{\omega_0}_{(\vec{a},0)+\varepsilon_I}(x_1,\dots,x_m;q)}}$$

Apply ω_0 on both sides, we have

both invariant under ω_0 b/c they are both symmetric

$\omega_0\left(\overline{e_\ell(x_1,\dots,x_{m-1})}\right) \ \|$

$$\frac{\prod_{\substack{1\le i\le j\le m\\ i+1<j}}\left(1-qt\frac{x_i}{x_j}\right)}{\prod_{1\le i<j\le m}\left(1-t\frac{x_i}{x_j}\right)}(x_1\cdots x_m)h_{N-m}(x_1,\dots,x_m)\overline{e_\ell(x_2,\dots,x_m)} = \sum_{\substack{(0,\vec{a}),\tau\in\mathbb{N}^m\\ |\tau|=N-m}}\ \sum_{\substack{I\subseteq[m-1]\\ |I|=\ell}} t^{|\vec{a}|} q^{d((0,\vec{a}),\tau)+h_I(\vec{a})}\,\omega_0\left(F^{\omega_0}_{(0,\vec{a})+\underbrace{(1,1,\dots,1)}_{m}+\tau}(x_1,\dots,x_m;q)\overline{E^{\omega_0}_{(\vec{a},0)+\varepsilon_I}(x_1,\dots,x_m;q)}\right).$$

$\square$

Extended Delta Conjecture (in §6.1)

$$H^m_q\left(\frac{\prod_{\substack{1\le i<j\le m \text{ s.t. }\\ i+1<j}}\left(1-qt\frac{x_i}{x_j}\right)}{\prod_{1\le i<j\le m}\left(1-t\frac{x_i}{x_j}\right)}x_1x_2\cdots x_m h_{N-m}(x_1,\dots,x_m)\overline{e_\ell(x_2,\dots,x_m)}\right)_{pol} = \sum_{\substack{J\subseteq[m-1]\\ |J|=\ell}}\ \sum_{\substack{\tau,(0,\vec{a})\in\mathbb{N}^m\\ |\tau|=N-m}} t^{|\vec{a}|} q^{d((0,\vec{a}),\tau)+h_J(\vec{a})}(\omega N_{\beta/\alpha})(X;q)$$ where $\beta=(0,\vec{a})+\underbrace{(1,1,\dots,1)}_{m}+\tau,\ \alpha=(\vec{a},0)+\varepsilon_J.$

Proof: Apply H^m_q on both sides of Theorem 6.3.6, take the polynomial parts, we get

By Path Prop 4.4.2, this is $\mathcal{L}^{\omega_0}_{((0,\vec{a})+\underbrace{(1,\dots,1)}_{m}+\tau)/(\vec{a},0)+\varepsilon_I}(x_1,\dots,x_m;q)_{pol}$

which is also $\omega N_{((0,\vec{a})+\overbrace{(1,\dots,1)}^{m}+\tau)/((0,\vec{a})+\varepsilon_I)}(x_1,\dots,x_m;q)=\omega N_{\beta/\alpha}(x_1,\dots,x_m;q)$

by Prop 6.2.2.

$$H^m_q\left(\frac{\prod_{\substack{1\le i<j\le m \text{ s.t. }\\ i+1<j}}\left(1-qt\frac{x_i}{x_j}\right)}{\prod_{1\le i<j\le m}\left(1-t\frac{x_i}{x_j}\right)}x_1x_2\cdots x_m h_{N-m}(x_1,\dots,x_m)\overline{e_\ell(x_2,\dots,x_m)}\right)_{pol} = \sum_{\substack{(0,\vec{a}),\tau\in\mathbb{N}^m\\ |\tau|=N-m}}\ \sum_{\substack{I\subseteq[m-1]\\ |I|=\ell}} t^{|\vec{a}|} q^{d((0,\vec{a}),\tau)+h_I(\vec{a})} H^m_q\left(\omega_0\left(F^{\omega_0}_{(0,\vec{a})+\underbrace{(1,1,\dots,1)}_{m}+\tau}(x_1,\dots,x_m;q)\overline{E^{\omega_0}_{(\vec{a},0)+\varepsilon_I}(x_1,\dots,x_m;q)}\right)\right)_{pol}$$

change variable I to J

$$= \sum_{\substack{J\subseteq[m-1]\\ |J|=\ell}}\ \sum_{\substack{\tau,(0,\vec{a})\in\mathbb{N}^m\\ |\tau|=N-m}} t^{|\vec{a}|} q^{d((0,\vec{a}),\tau)+h_J(\vec{a})}\,\omega N_{\beta/\alpha}(x_1,\dots,x_m;q).$$

$\square$